PH.D. SUBJET: STABILIZATION AND CONTROLLABILITY OF STEFAN'S PROBLEM

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ABSTRACT. In this Ph.D. subject, we will focus on the controllability and stabilization problem of the fusion interface of a solid. This problem can be modeled by Stefan's equations. These equations involve two heat equations coupled at the melting point. This is a non-linear problem involving free boundary.

Keywords: Control, Stabilization, Heat equation, Stefan's problem, Constrained state.

1. General issue, Context

In dimension one, the fusion or solidification of a solid can be represented by a Stefan problem. For a more general presentation of Stefan's problem, we can refer to [4, 9, 10, 7]. More precisely, suppose that the interval $(0,1)$ corresponds to the domain occupied by the liquid and solid phases. We will also assume that the two phases are connected, so we denote $s(t) \in (0,1)$ the position of the liquid-solid interface. We will also assume that the liquid phase occupies the interval $(0, s(t))$ and that the solid phase, the interval $(s(t), 1)$.

Denote by $y_l(t,x)$ the temperature in the liquid phase and by $y_s(t,x)$ the one in the solid phase and suppose that the melting temperature is null. The temperatures in the two phases are governed by the heat equation,

$$\begin{aligned} \dot{y}_l(t,x) &= \partial_x^2 y_l(t,x) && (t > 0, x \in (0, s(t)), \\ \dot{y}_s(t,x) &= \partial_x^2 y_s(t,x) && (t > 0, x \in (s(t), 1). \end{aligned}$$

At the liquid-solid interface, the temperature is equal to the melting temperature,

$$y_l(t, s(t)) = y_s(t, s(t)) = 0 \qquad (t > 0).$$

The motion of the interface $s(t)$ is governed by the ordinary differential equation,

$$\dot{s}(t) = -\partial_x y_l(t, s(t)) - \partial_x y_s(t, s(t)) \qquad (t > 0).$$

Finally, in order to avoid the appearance of new phases in the domain, we will impose that the solution satisfies,

$$y_l(t,s) \geqslant 0 \quad (x \in (0, s(t))) \quad \text{and} \quad y_s(t,s) \leqslant 0 \quad (x \in (s(t), 1)) \qquad (t > 0).$$

Finally, it will be possible to act on the state of the system using controls. As example, let us consider two controls $u_0(t)$ et $u_1(t)$ and impose the Neumann boundary conditions,

$$\partial_x y_l(t, 0) = u_0(t) \quad \text{and} \quad \partial_x y_s(t, 1) = u_1(t) \qquad (t > 0).$$

Of course, all of these equations are subject to initial conditions. Concerning the well-posed character of this system, one can refer to [2, 3].

2. Ph.D. Objectives

In recent work, [5, 6], the stabilization to trivial stationary states $\bar{y}_l = \bar{y}_s = 0$ et $s = \bar{s} \in (0,1)$ has been proved. This result was obtained using the baskstepping method, cf. [1]. However, this state feedback control requires knowledge of the entire state at all times. This constraint is not feasible in practice. This leads to a first question.

Q.1. Is it possible to stabilize the system towards a trivial stationary state, with a control depend only on a partial measure of the state?

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It is also natural to be interested in the stabilization of the system towards non-trivial stationary states. More precisely, the stationary states are decried by the following set of equations. Given two constants $\bar{u}_0$ et $\bar{u}_1$, the stationary states are the triplets $(\bar{y}_l, \bar{y}_s, \bar{s})$ solutions of:

$$\begin{gathered} 0 = \partial_x^2 \bar{y}_l(x) \quad (x \in (0, \bar{s})) \quad \text{and} \quad 0 = \partial_x^2 \bar{y}_s(x) \quad (x \in (\bar{s}, 1)), \\ \bar{y}_l(\bar{s}) = \bar{y}_s(\bar{s}) = 0, \\ \partial_x \bar{y}_l(0) = \bar{u}_0 \quad \text{and} \quad \partial_x \bar{y}_s(1) = \bar{u}_1. \\ 0 = -\partial_x \bar{y}_l(\bar{s}) - \partial_x \bar{y}_s(\bar{s}), \end{gathered}$$

Q.2. Is it possible to stabilize the system to a non-trivial steady state?

Because of the comparison principle for the heat equation, it is easy to show that any trivial stationary state cannot be reached in a finite time.

Q.3. Is it possible to control the system to a non-trivial steady state?

In [6], the constructed control ensures the sign conditions on the temperatures in the liquid and solid phases. This allows these state constraints to be omitted. However, these must be taken into account with other control strategies. In particular if the steady state is controllable, these positivity constraints will induce a minimum controllability time (cf. [8]).

Q.4. If is a steady state is controllable can we give a characterization of the minimum controllability time, and is there a control in this minimum time?

Finally, as far as we know, the only stabilization results relate to Stefan's unidimensional system. It is natural to wonder if similar results exist in the higher dimension. In this case, a major difficulty is to correctly formalize the control problem. Indeed, in dimension three (respectively two), the solid-liquid interface is a surface (respectively a curve). A simple way to approach this problem is to consider axially symmetrical liquid and solid domains. This allows coming back to a one-dimensional system. The previous questions arise again with the classical Laplace operator (∂_x^2) being replaced by the cylindrical or spherical Laplace operator.

3. Prerequisites

This subject requires both automatic and applied mathematics skills. In addition to basic knowledge of automatics, notions on partial differential equations will be desirable.

In addition, numerical experiments will be requested. It will therefore be necessary to have elementary notions in numerical approximation as well as to master numerical calculation software such as `Matlab` or `Scilab`.

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