MASTER 2 INTERNSHIP: SPARSE CONTROL

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Abstract. In recent decades, sparsity has been used extensively, especially for image compression. More recently, sparse control has been used for the control or stabilization of dynamic systems. The objective of this thesis is to further explore the possibility of sparse control for finite dimensional control systems. Attention will also be paid to the numerical computation of such controls.

1. Problematic and general context

The search for sparse solutions has experienced a boom in the last decades, especially for image processing, cf. e.g. [12]. More recently, sparse approaches have been proposed for the controllability of multi-agent systems, cf. e.g. [1, 2, 13]. Let us also note that, in the presence of a unilateral constraint on the control, a minimal time control is naturally sparse, cf. [8]. More generally, one can refer to the article [11].

Let us consider the linear control system of finite dimension,

$$\dot{x} = Ax + Bu, \qquad x(0) = x^0. \tag{$\star$}$$

We note m the dimension of the control u and n the dimension of the state x. We assume that the system ($\star$) is controllable, that is, for any time $T > 0$, any $x^0, x^1 \in \mathbb{R}^n$, there exists a control $u : [0, T] \to \mathbb{R}^m$ such that the solution $x(t) = x(t; u, x^0)$ of ($\star$) satisfies $x(T) = x^1$. Such a control can for example be obtained by minimizing its norm, typically, the L^2-norm. In other words, it is usual to look for the solution control of the following optimization problem:

$$(P_2) \qquad \mathcal{J}_2 = \min \left| \begin{array}{l} \int_0^T |u(t)|^2 \, \mathrm{d}t \\ u \in L^2(0,T)^m, \\ x(T; u, x^0) = x^1. \end{array} \right.$$

Such an approach leads to the HUM method (Hilbert Uniqueness Method), cf. [3]. On the other hand, the support of the obtained control is (except if $x^1 = e^{TA}x^0$) the whole interval $[0, T]$.

The objective of a sparse control is to minimize the action time of the control. Formally, the objective is to solve the minimization problem:

$$(P_0) \qquad \mathcal{J}_0 = \inf \left| \begin{array}{l} \int_0^T \chi(|u(t)|) \, \mathrm{d}t \\ u \in L^2(0,T)^m, \\ x(T; u, x^0) = x^1, \end{array} \right.$$

where $\chi(s) = 1$ if $s \neq 0$ and $\chi(0) = 0$. A result borrowed from [6], ensures that the controllability of a linear system can be done with a finite number of Dirac masses. Thus, this minimization problem does not admit a minimizer in L^2, but can admit one in the space of measures.

Another concern, in this minimization problem, is that the functional, $u \mapsto \|u\|_{L^0} := \int_0^2 \chi(|u(t)|) \, \mathrm{d}t$ is not convex. An approximate convex problem, which in some cases is equivalent to the original problem, cf. [4, 7, 11], is given by:

$$(P_1) \qquad \mathcal{J}_1 = \inf \left| \begin{array}{l} \int_0^T |u(t)| \, \mathrm{d}t \\ u \in L^2(0,T)^m, \\ x(T; u, x^0) = x^1. \end{array} \right.$$

In other words, the pseudo-norm L^0 is replaced by the norm L^1. As before, this minimization problem does not admit in general a minimizer in L^1. On the other hand, it admits a minimizer in the set of Radon measures.

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2. Goals

The objective of this thesis is to better understand the sparse structure of such controls, to extend the results to the nonlinear setting, and to propose efficient numerical methods leading to sparse controls. It will therefore be appropriate to focus on:

- the extension to the nonlinear framework of the results briefly mentioned in the previous paragraph. For this purpose, a typical model will be the Heisenberg system. In the nonlinear framework, it is also necessary to understand the impact of an impulsive control. For this, we refer to the work done in [10];
- the existence of a Lavrentiev jump. More precisely, do we have equality between the infimum for L^1 controls and the infimum for measure controls? For that, we can take inspiration from [9];
- the location of the control support. More precisely, can the obtained control be expressed as an event triggered control?
- the construction of numerical methods leading to sparse controls. For that, one can be inspired by greedy algorithms based on the Bregman distance, cf. [7, 12], or on the reformulations proposed in [5].

This list of problems is of course not exhaustive. The student may for instance also be interested in the stabilization problem with sparse controls.

3. Requirements for internship

A Master 2 in research, or equivalent, is required to apply for this thesis. This thesis subject requires skills in automatic control/applied mathematics. In addition to mastering the basics of automation, a background in optimal control and numerical computation will be desirable.

If you are interested in this subject, please send your application to Marc Jungers and Jrôme Lohac (emails: `marc.jungers@univ-lorraine.fr` and `jerome.loheac@univ-lorraine.fr`).
The documents required for the application are:

- a CV, mentioning in particular if you have already obtained your Master 2 degree, or if you are going to obtain it before this summer, as well as the ranking you have obtained with your last diplomas;
- last transcripts;
- a copy of your passport indicating your date of birth, your place of birth and a current address;
- a cover letter;
- a few letters of recommendation (sent directly to us) or contact person.

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