Opinion dynamics control by leadership with bounded influence

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***Abstract*— In the present paper, we provide results on the control of general opinion dynamics systems. The control is applied to one agent only, called the leader. Explicit control laws ensure three complementary desired behaviours of the system: i) drag all agents arbitrarily close to the leader, then ii) make all agents follow the leader toward a targeted value, iii) make all agents converge to a targeted consensus value. Unlike the existing literature, the control is carried out under weak assumptions on the leader influence. In particular, the leader may only have a bounded influence range (this for instance includes the Hegselmann-Krause bounded confidence model). Finally our results are illustrated by numerical examples.**

I. INTRODUCTION

Modelling social systems using engineering tools such as multi-agent systems is a recent and promising endeavour. Multi-agent systems have been shown to efficiently model how opinion dynamics occur as a result of social interactions [1]. Models of opinion dynamics are often based on the consensus system, which captures the process of agreement among a group of agents. In the past decade, the consensus system has been thoroughly analysed, yielding a number of conditions on the strength of interactions guaranteeing that the agents reach consensus [2]–[10]. Other works have also studied the formation of opinion clusters [11], [12].

The existing literature on opinion dynamics systems has mainly focused on the analysis of autonomous systems. Only few studies have tackled the possibility of adding external stimuli in order to drag the systems to a desired target state. However, such interventions are widespread in social systems and significantly impact our markets, politics [13], [14] and health. External stimuli may for instance come in the form of advertisements or State subsidies. Among others, it may help to promote the adoption of innovative behaviours (e.g., quit smoking [15], eat healthy) [16].

Understanding the impact of these external stimuli on opinion dynamics systems is critical to plan successful interventions in social networks. The present work describes such a study. Other recent approaches have been proposed to intervene in social systems. In [17], the authors define centrality measure assessing the ability of a node to control the system. However, there, the dynamics of the social system is described by a generic system of linear differential equations. Instead, [14], [18] and [19] have studied the possibility to control opinion dynamics systems such as the Hegselmann-Krause bounded confidence model [20]. These studies usually assume that one or several agents, called the *leaders* (or *strategic agents*), are not influenced by the others. Instead, the dynamics of the leaders evolve following a control input. Related pieces of work are [21], [22], where the authors designed the optimal control for Cucker-Smale flocking systems.

The systems studied in [14] are the discrete-time Hegselmann-Krause bounded confidence model and the discrete-time linear DeGroot model. The authors use one leader to drag as many agents as possible in finite time to a target opinion interval. In [18], the author uses leaders in order to decrease the time before the states of all agents have converged. The study is carried out on the discrete-time Hegselmann-Krause bounded confidence model.

In the present work, the goal of the leader is to make all agents converge toward a specific target value. To the best of our knowledge, the only existing work which studies the control of opinion dynamics to drag all the agents toward a target consensus value is [19]. Like ours, the study focuses on a continuous-time model. Consistently with the literature on opinion dynamics, the influence function depends on the distance between opinions. However, the leader influence function is assumed to remain positive for arbitrarily large opinion distance. This assumption does not hold for instance in the bounded confidence influence model and it requires the leader to have more influence than the rest of the agents. This may not always be a realistic assumption. In the present work, we show that a weaker assumption suffices to control general opinion dynamics systems.

Precisely, we show that a general model of opinion dynamics, in the presence of one leader, can be controlled with weak assumptions on the leader influence function. In particular, we do not assume that the leader has a larger influence range or higher influence strength than the other agents. On the contrary, our results allow an influence function with bounded support, i.e., where influence vanishes if the distance between opinions is too large. Finally, we show that a leader whose speed is saturated still achieves the control of the opinion dynamics. Both the amplitude of the saturation and the maximal influence range may be chosen arbitrarily small. These bounds may represent natural economic or media limitations of the leader's power.

To make all agents converge toward a specific target value, we design a control which acts in three steps. First, gather all the agents near the leader in finite time. A bound on this time is provided ; it depends on the parameters selected in the control. When all agents are sufficiently close to the leader, drag the agents toward a desired opinion, in finite time. A time bound is also provided here. When the leader

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reaches this desired opinion, make all agents tend toward a consensus at the desired opinion.

This paper is organized as follows. In Section II, we define the multi-agent opinion dynamics system we study and we describe the problem dealt with in the sequel. Section III contains preliminary lemmas leading to the main results. In Section IV, we make explicit the control law which drives all the agents toward a consensus at the desired target opinion. Section V presents numerical simulations of the system to illustrate the results of this paper. Finally, we conclude in Section VI.

Throughout the paper, the notation $\mathbb{R}$ will denote the set of real numbers, $\mathbb{R}^+$ the set of the non-negative real numbers and the notation $|.|$ will denote the absolute value for a real number.

II. PROBLEM FORMULATION

A. System definition

Let us consider a system of $n+1$ agents, numbered from 0 to n and forming a multi-agent system in continuous time. The opinion or state of agent i at time $t \in \mathbb{R}^+$ is a scalar denoted $x_i(t) \in \mathbb{R}$. Among the $n+1$ agents, we distinguish n uncontrolled agents and one controlled agent called leader indexed by 0. In this paper, the leader is not influenced by the other agents of the system, but it can influence other agents. The dynamic of the multi-agent system is then given, for $i \in \mathcal{N} \triangleq \{1, \dots, n\}$ and $t \in \mathbb{R}^+$ by

$$\left\{\begin{array}{rcl} \dot{x}_i(t) & = & \sum\limits_{j=1}^{n} f(|x_j(t)-x_i(t)|)(x_j(t)-x_i(t)) \\ & & +\ f_0(|x_0(t)-x_i(t)|)(x_0(t)-x_i(t)), \\ \dot{x}_0(t) & = & u(t),\ |u(t)| \leq \mu,\ \mu > 0\ , \end{array}\right. \tag{1}$$

where f and f_0 are non-negative piecewise continuous influence functions defined on $\mathbb{R}^+$ and where $u(t)$ is a control on the leader's opinion $x_0(t)$. The control is bounded in norm by a natural saturation denoted μ. This model allows the leader to influence the agents with an influence function which can be different from the influence function ruling the interactions between agents. In the sequel we will assume, for a given set of initial conditions, the existence of a unique solution to (1) on $\mathbb{R}^+$. We call this solution the trajectory of the system. The question of existence and uniqueness of this solution for general functions f and f_0 is a hard problem. The problem was studied in [23] for the particular case of the Hegselmann-Krause bounded confidence model.

Only the following condition on the influence function of the leader f_0 is assumed. This means that no assumption is made on the influence function f of the agents, other than being piecewise continuous.

Assumption 1: It exists $\eta > 0$ such that $\forall\, y \in [0;\eta]$ we have $f_0(y) \geq f_0(\eta)$.

We say that an agent i is in the η *influence zone* of the leader at time t whenever $|x_i(t) - x_0(t)| \leq \eta$. In this case, $f_0(|x_i(t)-x_0(t)|) \geq f_0(\eta)$. Assumption 1 then ensures a lower bound on the influence of the leader within an η influence zone.

Because of the presence of the leader, system (1) is not autonomous. For this reason, the notion of consensus has to be clarified. Indeed we can distinguish two kinds of agreements, each being detailed in the following definitions.

Definition 1: We call *consensus* a synchronisation among the agents not including the leader, that is

$$\forall\,(i,j) \in \mathcal{N}^2,\ \lim_{t\to+\infty} |x_i(t) - x_j(t)| = 0.$$

Definition 2: We call *leader agreement* an agreement among the agents and the leader

$$\forall\, i \in \mathcal{N},\ \lim_{t\to+\infty} |x_i(t) - x_0(t)| = 0.$$

From these definitions we can see that leader agreement implies consensus.

As every agent is governed by the same dynamics, the order of the agents remains the same, meaning that

$$\forall\,(i,j) \in \mathcal{N}^2,\ x_i(0) \geq x_j(0) \implies \forall\, t \geq 0,\ x_i(t) \geq x_j(t).$$

This holds because when two agents share the same opinion, (1) ensures that both agents then share the same dynamics. Thus, the order between the agents remains the same at all time. For the rest of the paper, without loss of generality, we relabel the agents in increasing order of opinions, that is $\forall\, t \geq 0$, $x_1(t) \leq x_2(t) \leq \cdots \leq x_n(t)$.

B. Problem statement

In this paper we focus on presenting a control satisfying the following properties.

Problem 1: Given an arbitrary consensus value $\alpha \in \mathbb{R}$ and arbitrary fixed initial conditions $x_i(0) \in \mathbb{R}$, for $i \in \mathcal{N}$ and $x_0(0) \in \mathbb{R}$. Find a control $u(t)$, for $t \in \mathbb{R}^+$ and with $|u(t)| < \mu$, $\mu > 0$ such that all x_i converge toward α, i.e. $\lim\limits_{t\to+\infty} x_i(t) = \alpha$.

The problem we just stated is subdivided in the three following subproblems, which are addressed separately in this paper. Each subproblem focuses on finding a control ensuring that certain features are obtained. Concatenating the solutions of these three subproblems leads to a solution of Problem 1.

Subproblem 1: Given abitrary fixed initial conditions, find a finite time T and a control $u(t)$ with $|u(t)| < \mu$, $\mu > 0$ for $t \in [0,T]$, such that $\forall\, i \in \mathcal{N},\ |x_i(T) - x_0(T)| < \eta$.

The first subproblem deals with the possibility of rallying all the agents in the η influence zone of the leader in finite time, disregarding the value around which the agents are gathered. The goal of the second subproblem is to drag all the gathered agents toward a desired opinion here called target value.

Subproblem 2: Let $t_0 \in \mathbb{R}^+$. Given a target consensus value $\alpha \in \mathbb{R}$ and arbitrary fixed initial conditions such that $\forall\, i \in \mathcal{N},\ |x_i(t_0) - x_0(t_0)| < \eta$, find a time $t_1 \geq t_0$

and a control $u(t),\ t \in [t_0, t_1]$, such that $\forall\, i \in \mathcal{N}$, we have $|x_i(t_1) - x_0(t_1)| < \eta$, and $x_0(t_1) = \alpha$.

To completely address Problem 1, once the leader is at the target opinion, we have to ensure that all agents converge toward the desired consensus value.

Subproblem 3: Let $t_0 \in \mathbb{R}^+$. Given a target consensus value $\alpha \in \mathbb{R}$ and fixed initial conditions such that $x_0(t_0) = \alpha$ and $\forall\, i \in \mathcal{N},\ |x_i(t_0) - x_0(t_0)| < \eta$. Find a control $u(t)$ with $|u(t)| < \mu$, $\mu > 0$, $t \in [t_0, +\infty)$, such that all agents converge and $\lim_{t\to+\infty} x_i(t) = \alpha$.

III. PRELIMINARIES

In this section we present general lemmas which will serve as a basis for the forthcoming results.

The following lemma describes the behaviour of the extreme agents in the particular case where the leader is amidst the other agents.

Lemma 1: Let $t \in \mathbb{R}^+$. If $x_0(t) \geq x_1(t)$ then we have,

$$\dot{x}_1(t) \geq f_0(|x_0(t) - x_1(t)|)(x_0(t) - x_1(t)) \geq 0. \quad (2)$$

Likewise if $x_0(t) \leq x_n(t)$, then we have

$$\dot{x}_n(t) \leq -f_0(|x_0(t) - x_n(t)|)(x_0(t) - x_n(t)) \leq 0. \quad (3)$$

This means that when the leader is above the lowest agent, the lowest agent has a non-decreasing trajectory. Likewise when the leader is below agent n, the highest agent, agent n has a non-increasing trajectory. Intuitively, when no agent or the leader is below agent 1, then agent 1 is not influenced by any agent with a lower opinion, thus it cannot decrease.

If the leader is amidst the agents, then both (2) and (3) apply and we can deduce that the agents tend to get closer. This idea will help to prove Theorem 3 which deals with the convergence toward consensus and solve Subproblem 3.

Proof: We prove the result for agent 1. The result for agent n is obtained by the same reasoning. Let us suppose that for some $t \in \mathbb{R}^+$ we have $x_0(t) \geq x_1(t)$. As the order between the agents is preserved, agent 1 then has the lowest opinion in value and we have $\forall\, i \in \mathcal{N},\ (x_i(t) - x_1(t)) \geq 0$, from which we deduce

$$\sum\nolimits_{i=1}^{n} f(|x_i(t) - x_1(t)|)(x_i(t) - x_1(t)) \geq 0,$$

as function f is non-negative. From (1) we then get the conclusion of the lemma because $x_0(t) \geq x_1(t)$ and function f_0 is non-negative. ■

The following lemma ensures that if the leader's opinion is below all the opinions of the agents, and if the control is sufficiently small in norm, then the leader is able to force the agent having the lowest opinion to increase its opinion as much as wanted.

Lemma 2: Under Assumption 1, if

$$\exists\, t_0 \geq 0,\ x_1(t_0) > x_0(t_0) - \eta, \quad (4)$$

and if

$$\forall\, t \geq t_0,\ |u(t)| \leq \eta\, f_0(\eta), \quad (5)$$

then

$$\forall\, t \geq t_0,\ x_1(t) > x_0(t) - \eta. \quad (6)$$

Proof: Let us show that (6) is true by contradiction. In that case, by continuity of the trajectory, it exists an instant

$$t_2 = \min\{\tau \geq t_0 \mid x_1(\tau) = x_0(\tau) - \eta\},$$

where the agent 1 is at the frontier of the η influence zone of the leader. By continuity of the trajectories, we can also define an instant $t_1 \in [t_0, t_2)$, sufficiently close to t_2, such that $\forall\, t \in [t_1, t_2],\ x_1(t) < x_0(t)$, which ensures

$$\forall\, t \in [t_1, t_2],\ x_1(t) \in [x_0(t) - \eta, x_0(t)). \quad (7)$$

For $t \in [t_1, t_2]$, equation (2) from Lemma 1 applies as the leader is above agent 1. Moreover, (7) and Assumption 1 yield $f_0(|x_0(t) - x_1(t)|) \geq f_0(\eta)$, which allows to deduce from (2) $\dot{x}_1(t) + f_0(\eta)\, x_1(t) \geq f_0(\eta)\, x_0(t)$. To analyse this differential inequality, we consider the associated differential equation $\dot{\widetilde{x}}_1(t) + f_0(\eta)\ \widetilde{x}_1(t) = f_0(\eta)\ x_0(t)$, with the initial condition $\widetilde{x}_1(t_1) = x_1(t_1)$. Then, the trajectory $\widetilde{x}_1(t)$ satisfies $\forall\, t \in [t_1, t_2],\ x_1(t) \geq \widetilde{x}_1(t)$. As a consequence, (4) gives $\widetilde{x}_1(t_1) > x_0(t_1) - \eta$. Let us show by contradiction

$$\forall\, t \in [t_1, t_2],\ \widetilde{x}_1(t) > x_0(t) - \eta. \quad (8)$$

To achieve this, we suppose the existence of

$$\widetilde{t}_2 = \min\{\tau \in [t_1, t_2] \mid \widetilde{x}_1(\tau) = x_0(\tau) - \eta\}.$$

We then study the evolution of the quantity

$$z(t) = \widetilde{x}_1(t) - (x_0(t) - \eta),$$

for $t \in [t_1, \widetilde{t}_2]$. We then have

$$\begin{aligned} \dot{z}(t) &= \dot{\widetilde{x}}_1(t) - \dot{x}_0(t) \\ &= f_0(\eta)\big(x_0(t) - \widetilde{x}_1(t)\big) - \dot{x}_0(t) \\ &= f_0(\eta)\big(\eta - z(t)\big) - \dot{x}_0(t). \end{aligned}$$

Since by (5) for $t \in [t_1, \widetilde{t}_2]$, $\dot{x}_0(t) \leq \eta\ f_0(\eta)$, we have the inequality $\dot{z}(t) \geq -f_0(\eta)\ z(t)$. Notice that by (8) we also have $z(t_1) = \widetilde{x}_1(t_1) - (x_0(t_1) - \eta) > 0$. The Gronwall Lemma from [24] then ensures

$$z(t_2) \geq z(t_1)\ \exp\big(-f_0(\eta)\,(t_2 - t_1)\big) > 0.$$

This contradicts the definition of $\widetilde{t}_2$, thus neither $\widetilde{t}_2$ nor t_2 exist and the lemma is proven. ■

By considering the trajectories $x_i := -x_i$, we can infer the following corollary from Lemma 2.

Corollary 1: Under Assumption 1, if $\exists\, t_0 \geq 0$ such that $x_n(t_0) < x_0(t_0) + \eta$, and if $\forall\, t \geq t_0,\ |u(t)| \leq \eta\, f_0(\eta)$, then $\forall\, t \geq t_0,\ x_n(t) < x_0(t) + \eta$.

IV. MAIN RESULTS

This section contains three theorems containing the main results of this paper. Each theorem deals with one of the subproblems stated in II-B and thus the three following theorems address Problem 1.

A. Rallying control

Theorem 1: Let $c \in (0, \sigma]$, κ in $(0, \min\{\eta f_0(\eta), \mu\}]$ and $\varepsilon \in (0, \eta)$. Under Assumption 1, for any initial conditions $x_i(0) \in \mathbb{R}$, for $i \in \mathcal{N}$ the following control

$$u(t) = \begin{cases} -c & \text{for } t \in [0, T_1), \\ \kappa & \text{for } t \in [T_1, T_2], \end{cases} \tag{9}$$

where

$$T_1 = \inf\{t \geq 0 \mid x_0(t) \leq x_1(t) + \varepsilon\}, \tag{10}$$

and

$$T_2 = \inf\{t \geq T_1 \mid x_0(t) \geq x_n(t) - \varepsilon\}, \tag{11}$$

solves Subproblem 1.

The control we propose is divided into two steps: first the leader reaches the agent having the lowest opinion, and then it attracts this agent toward the one having the highest opinion, thus gathering all agents near himself. To prove Theorem 1, we need Lemmas 3 and 4.

Remark 1: The choice of a value for $\varepsilon \in (0, \eta)$ is arbitrary. It is important that $\varepsilon < \eta$, this will be made explicit in the second step of the control.

Lemma 3: Instant T_1 defined in (10) is finite with

$$T_1 \leq T_1' \triangleq \max\left\{0, \frac{x_0(0) - (x_1(0) + \varepsilon)}{c}\right\}. \tag{12}$$

Moreover, we have $x_0(T_1) \leq x_1(T_1) + \varepsilon$.

Proof: To show that $T_1 \leq T_1'$ let us first remark that if $T_1' = 0$ then $T_1 = 0$. Then we suppose, to obtain a contradiction, that $T_1 > T_1' > 0$. The definition of T_1 in (10) then allow us to write $\forall\, t \in [0, T_1),\ x_0(t) > x_1(t) + \varepsilon$.

Equation (2) from Lemma 1 ensures that $x_1(t)$ is non-decreasing for $t \in [0, T_1)$. Since $T_1' < T_1$, $x_1(t)$ is also non-decreasing for $t \in [0, T_1']$, yielding $x_1(0) \leq x_1(T_1')$. Then according to (12) and by integration of (9)

$$x_0(T_1') = x_0(0) - c\, T_1' = x_1(0) + \varepsilon \leq x_1(T_1') + \varepsilon,$$

which contradicts (12) and then proves the result and the lemma. ∎

The second step of the control aims at dragging the lowest agent toward the highest. To achieve this, the leader's opinion has to increase sufficiently slowly in order to ensure that the lowest agent does not fall below the η influence zone of the leader.

Lemma 4: Under Assumption 1, instant T_2 defined by (11) is finite and

$$T_2 \leq T_2' \triangleq T_1 + \max\left\{0, \frac{(x_n(T_1) - \varepsilon) - x_0(T_1)}{\kappa}\right\}, \tag{13}$$

with $\varepsilon \in (0, \eta)$. Moreover,

$$\forall\, i \in \mathcal{N},\ \forall\, t \in [T_1, T_2],\ x_i(t) > x_0(t) - \eta, \tag{14}$$

$$\forall\, i \in \mathcal{N},\ |x_i(T_2) - x_0(T_2)| < \eta. \tag{15}$$

Proof: The proof of (13) follows the same method used in the proof of Lemma 3 to prove (12). With $t_0 := T_1$, equation (4) from Lemma 2 is satisfied and the bound on κ in Theorem 1 ensures that (5) holds. As a consequence and thanks to Assumption 1, Lemma 2 applies. Recalling that the order of the agents remains the same, statement (14) is a direct consequence of Lemma 2. By definition of T_2, we have $x_0(T_2) = x_n(T_2) - \varepsilon$, which combined with (14) gives (15). ∎

The previous lemmas allow us to solve Subproblem 1 by proving Theorem 1.

Proof of Theorem 1: Applying Lemmas 3 and 4, we obtain the conclusion of Subproblem 1 with $T := T_2$. ∎

Remark 2: The control proposed by Theorem 1 allows to gather all agents as close as we want from the leader by tuning parameter η. Theorem 1 also guarantees the existence of a control law which solves Subproblem 1, regardless of the control saturation $\mu > 0$: the saturation amplitude can be chosen arbitrarily small.

B. Trajectory tracking and consensus

We then present a theorem giving conditions on the speed of the leader to ensure trajectory tracking without losing influence over the agents.

Theorem 2: Under Assumption 1, if the control is sufficiently small in norm and all agents are initially in the η influence zone of the leader, then all the agents remain at all time in the η influence zone of the leader. Formally, if at a time t_0 we have $\forall\, i \in \mathcal{N},\ |x_i(t_0) - x_0(t_0)| < \eta$, and if

$$\forall\, t \geq t_0,\ |u(t)| \leq \min\{\eta\, f_0(\eta), \mu\}, \tag{16}$$

then $\forall\, t \geq t_0,\ \forall\, i \in \mathcal{N},\ |x_i(t) - x_0(t)| < \eta$.

Proof: Lemma 2 ensures $\forall\, t \geq t_0,\ x_1(t) > x_0(t) - \eta$. Moreover, Corollary 1 ensures $\forall\, t \geq t_0,\ x_n(t) < x_0(t) + \eta$. As the order of the agents remains the same, the theorem is proven. ∎

Theorem 2 provides a direct way to solve Subproblem 2 as expressed in the next corollary.

Corollary 2: If $\alpha = x_0(t_0)$ in Subproblem 2, then it is solved with $T = t_0$. Otherwise, under Assumption 1, the following control law solves Subproblem 2

$$u(t) = \frac{\alpha - x_0(t_0)}{|\alpha - x_0(t_0)|} \min\{\eta f_0(\eta), \mu\}, \forall t \in [t_0, t_0 + T),$$

where $T = \dfrac{|\alpha - x_0(t_0)|}{\min\{\eta f_0(\eta), \mu\}}$.

Proof: Following the conditions in Subproblem 1, we start by assuming $\forall i \in \mathcal{N}, |x_i(t_0) - x_0(t_0)| < \eta$. It is clear that $\forall t \in [t_0, t_0 + T)$, $|u(t)| \in (0, \min\{\eta f_0(\eta), \mu\}]$ so that Theorem 2 applies. Moreover, by integration of (1) we have $x_0(T) = \alpha$, so that Subproblem 2 is solved. ∎

Once the leader has reached its target value α while keeping all agents in its η influence zone, it remains to find a control law which makes all agents converge to α to solve Subproblem 3. This is precisely what the next theorem provides.

Theorem 3: Under Assumption 1, if a null control is applied and all agents are initially in the η influence zone of the leader, then leader agreement is reached asymptotically and consensus is obtained. Formally, if at a time t_0 we have $\forall i \in \mathcal{N},\ |x_i(t_0) - x_i(t_0)| < \eta$, and if $\forall t \geq t_0,\ u(t) = 0$, then $\forall\, i \in \mathcal{N},\ \lim_{t \to +\infty} x_i(t) = x_0(t_0) = \alpha$.

Proof: Since (16) holds, Theorem 2 applies and then $\forall\, t \geq t_0, \forall\, i \in \mathcal{N},\ |x_i(t) - x_0(t)| < \eta$. Let us define the following instant

$$T_3 = \min\{t \geq t_0 \mid x_n(t) \leq x_0(t) \text{ or } x_1(t) \geq x_0(t)\}.$$

Two cases are possible:

First case, $T_3 < +\infty$ *:* then we will show that all agents remain on the same side of the leader after the instant T_3. Let us then suppose, without loss of generality, that $x_n(T_3) \leq x_0(T_3)$, the other case being symmetrical. We then show by contradiction that

$$\forall\, t \geq T_3,\ x_n(t) \leq x_0(t), \tag{17}$$

by supposing the existence of an instant $t_3 > T_3$ such that $x_n(t_3) > x_0(t_3)$. By continuity of the trajectory, it then exists $t_1 \in [T_3, t_3]$ such that $x_n(t_1) = x_0(t_1)$. Still by continuity, t_3 can be chosen close enough to t_1 such that

$$\forall\, t \in [t_1, t_3],\ x_n(t) \geq x_0(t).$$

On the one hand, $x_n(t_1) = x_0(t_1)$ and $x_n(t_3) > x_0(t_3)$. So by then mean value theorem, it exists $t_2 \in (t_1, t_3)$ such that $\dot{x}_n(t_2) > \dot{x}_0(t_2)$. On the other hand, $x_n(t_2) \geq x_0(t_2)$ so that by Lemma 1, $\dot{x}_n(t_2) \leq 0$, which contradicts the first fact and (17) is proven.

Let $t \geq T_3$. Since the leader is above agent 1, equation (2) from Lemma 1 applies. Knowing also that $\dot{x}_0(t) = 0$, we deduce from (2) that $\dot{x}_0(t) - \dot{x}_1(t) \leq -f_0(\eta)\big(x_0(t) - x_1(t)\big)$. Using the Gronwall Lemma we then get

$$0 \leq x_0(t) - x_1(t) \leq \big(x_0(T_3) - x_1(T_3)\big) \exp(-f_0(\eta)(t - T_3)),$$

and deduce $\lim_{t \to +\infty} x_0(t) - x_1(t) = 0$. As the order of the agents remains the same, by using (17) we obtain the conclusion of the theorem.

Second case, $T_3 = +\infty$ *:* the leader is between agents 1 and n at all times and Lemma 1 applies. By combining (2) and (3) we obtain $\dot{x}_n(t) - \dot{x}_1(t) \leq f_0(\eta)(x_n(t) - x_1(t))$. Using the Gronwall Lemma we get

$$0 \leq x_n(t) - x_1(t) \leq (x_n(t_0) - x_1(t_0)) \exp(-f_0(\eta)\,(t - t_0)),$$

which proves $\lim_{t \to +\infty} x_1(t) - x_n(t) = 0$. Since the leader remains between agents 1 and n at all times and the order between the agents stays the same, we obtain the conclusion of the theorem. ■

V. NUMERICAL ILLUSTRATIONS

In this section we present numerical simulations to depict the previous results.

We first illustrate the control we proposed in Theorems 1 and 3 to solve Subproblems 1 and 3. Consider a system composed of 6 agents and a leader. In this example we assume $f = 0$, meaning the agents do not interact and therefore are not attracted to each other. The leader has the following Hegselmann-Krause influence function from [20] on $\mathbb{R}^+$

$$f_0(y) = \begin{cases} 1 & \text{if } y \in [0, 1], \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

This means that the leader's influence range is 1. As a consequence, an agent is not influenced by the leader if the difference between its opinion and the opinion of the leader is greater than 1. In this example, Assumption 1 is satisfied taking $\eta = 1$. Note that in this case, Assumption 1 would hold taking any value of η in the interval $(0, 1]$. We set the saturation constant $\mu = 1.5$. The control law of the leader is divided into two steps. First, the control is set as described in Theorem 1 for $t \in [0, T_2]$, with T_2 being defined in Theorem 1. During this step the parameters of the control from Theorem 1 are set as follows. The speed $u(t) = -c$, with $c = \mu = 1.5$, is applied until the leader reaches agent 1 at time T_1 as defined in Theorem 1, with distance $\varepsilon = 0.5$. Then, the speed $u(t) = \kappa = \eta\, f_0(\eta) = 1$ is applied while the leader pulls agent 1 until reaching agent n at time T_2 as defined in Theorem 1. Lemma 2 ensures that since the control is bounded in norm by $\eta\ f_0(\eta)$, agent 1 follows the leader during this step. At time T_2, agent n is also inside the ε influence zone of the leader so that all the agents are inside the η influence zone of the leader and Subproblem 1 is solved. Once all agents are gathered around the leader, a null control is applied, i.e., $u(t) = 0$ for $t > T_2$, to reach consensus following Theorem 3. This shows how Subproblem 3 is solved. Figure 1 depicts trajectories of the multi-agent system where all agents are initially equally spaced in the opinion range $[0, 10]$ and the leader has an initial opinion of $x_0(0) = 5$.

In Figure 2 we illustrate Theorem 2 with the same multi-agent system used for the previous illustration, differing only in the initial conditions. All agents are here initially in the opinion range $(-1, 1)$ and the leader has an initial opinion of 0. Because all agents are within the η influence zone of the leader, the initial time satisfies the definition of T_2 given in Theorem 1. The control used on the leader is divided into three parts. Between initial time T_2 and time t_1, a sine of amplitude $\eta\, f_0(\eta)$ is applied. Then between time t_1 and t_2, a constant control of value $\eta\ f_0(\eta)$ is used. As the control is bounded in norm by $\eta\ f_0(\eta) = 1$, Theorem 2 applies and all agents stay in the η influence zone of the leader. Finally, after time t_2, the leader is given a control $u(t) > \eta\ f_0(\eta)$. Since Theorem 2 does not apply, there is no guarantee that all agents remain within the η influence zone of the leader. In this particular case, we observe that eventually all agents leave the η influence zone of the leader, but this may not always be the case.

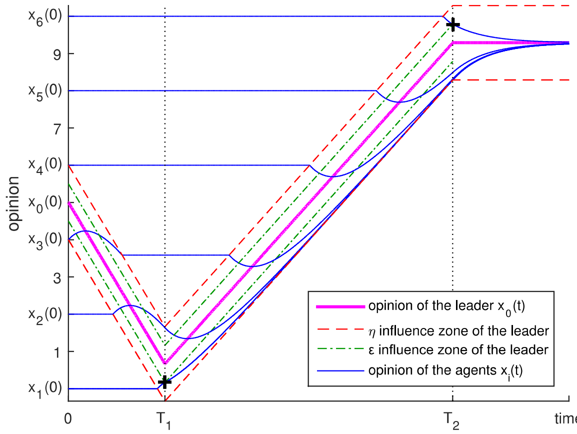

Fig. 1. Example of control law proposed in Theorems 1 and 3 applied to a multi-agent system. The opinions of the 6 agents are in blue while the opinion of the leader is in magenta. The η influence zone of the leader introduced in Assumption 1 is depicted by a dashed red line and the ε influence zone of the leader used in Theorem 1 is shown with a green dot-dashed line. The two black crosses emphasize the definition of instants T_1 and T_2 in Theorem 1.

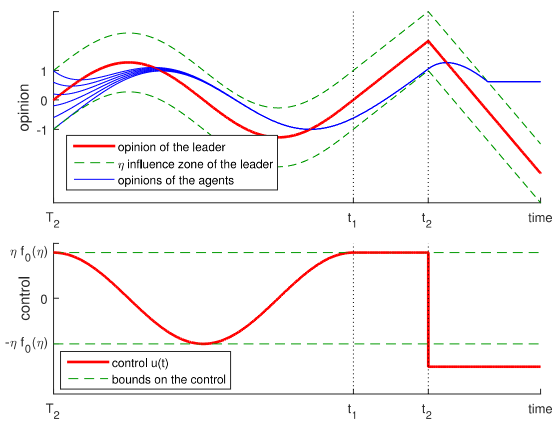

Fig. 2. Example of trajectory tracking of 6 agents following a leader to illustrate Theorem 2. On the top graph, the opinions of the agents are depicted by blue lines while the opinion of the leader is in red. The η influence zone of the leader defined by Assumption 1 is shown with a green dashed line. The bottom graph depicts the control $u(t)$ applied to the leader in red along with the bounds given by Theorem 2.

VI. CONCLUSION

In this paper we presented results on the control of opinion dynamics systems by proposing a control law acting on a specific agent called leader. The control law drives all the agents toward a target consensus value in three consecutive steps. First the leader gathers the agents around its opinion, then it drags them toward a target opinion value and finally ensures that all agents converge toward consensus. This control law applies even if the leader only has a bounded influence range and bounded speed. We finally presented numerical illustrations of the main results of the paper.

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