Sufficient Conditions for Flocking via Graph Robustness Analysis

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***Abstract*— In this paper, we consider a multi-agent system consisting of mobile agents with second-order dynamics. The communication network is determined by a standard interaction rule based on the distance between agents. The goal of this paper is to determine practical conditions (on the initial positions and velocities of agents) ensuring that the agents asymptotically agree on a common velocity (i.e. a flocking behavior is achieved). For this purpose, we define a new notion of graph robustness which allows us to establish such conditions, building upon previous work on multi-agent systems with switching communication networks. Though conservative, our approach gives conditions that can be verified a priori. Our result is illustrated through simulations.**

I. Introduction

Cooperative behaviors generating complex phenomena are observed in nature [1], [4]. Multi-agent systems also find applications in technical areas such as mobile sensor networks [5], cooperative robotics [3] or distributed implementation of algorithms [13]. A central question arising in the study of multi-agent systems is whether the group will be able to reach a consensus. Intuitively, agents are said to reach a consensus when all individuals agree on a common value (e.g. the heading direction of a flock of birds, the candidate to elect for voters).

To carry out formal studies on consensus problems, one usually assumes that the multi-agent system follows some abstract communication protocol and then investigates conditions under which a consensus will be reached. Existing frameworks include discrete and continuous-time systems, involving or neglecting time-delays in the communication process. The communication network between agents is usually modeled by a graph. Its topology is either assumed to be fixed, or can switch over time. The switching topology of the interactions is sometimes assumed to depend on the state of the agents (e.g. the strength of the communication can be a function of the distance between agents). The order of the dynamics of the agents also varies between the different models. For example, second-order models can be useful to represent the dynamics of both the speed and position of agents. Olfati-Saber, Fax and Murray review results on the subject in [10].

Most papers have investigated sufficient conditions ensuring asymptotic consensus. The assumptions made in the models are usually rather general (see e.g. [9]). This enables the given conditions to apply in a wide range of cases. Conditions usually require invariant connectivity properties in the communication network over time. A drawback in such conditions is that they often cannot be verified a priori.

In this paper we consider a group of agents with second-order dynamics. The communication network is determined by a standard interaction rule based on the distance between agents. The goal of this paper is to determine practical conditions (on the initial positions and velocities of agents) ensuring that the agents eventually agree on a common velocity (i.e. a flocking behavior is achieved). We define a new notion of graph robustness which allows us, building upon previous work such as [11], to establish such conditions. Though conservative, our approach gives conditions that can be verified a priori. Moreover, it is computationally tractable and can be fully automated. Our result is illustrated through simulations.

Another paper has investigated sufficient conditions on the initial positions and velocities ensuring flocking [7]. However, the conditions in [7] ensure that the initial communication links are preserved for all time whereas our more general conditions allow some of the initial communication links to be broken, provided a path is preserved between the agents.

II. Problem Formulation

In this paper, we study a continuous-time, multi-agent system. We consider a set $\mathcal{V} = \{1, \dots, n\}$ of mobile *agents* evolving in a d-dimensional space. Each agent $i \in \mathcal{V}$ is characterized by its position $x_i(t) \in \mathbb{R}^d$ and its velocity $v_i(t) \in \mathbb{R}^d$. The initial positions and velocities are given by $x_i(0) = x_i^0$ and $v_i(0) = v_i^0$. The agents exchange information over a *communication network* given by a graph $G(t) = (\mathcal{V}, \mathcal{E}(t))$; the topology of the communication network depends on the relative position of agents and is therefore subject to change. The agents use the available information to adapt their velocity in order to achieve a flocking behavior.

Formally, the evolution of each agent $i \in \mathcal{V}$ is described by the following system of differential equations:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= \sum_{j \in \mathcal{N}_i(t)} (v_j(t) - v_i(t)) \end{aligned} \tag{1}$$

where $\mathcal{N}_i(t) = \{j \in \mathcal{V};\ (j,i) \in \mathcal{E}(t)\}$ is the set of agents communicating with i at time t, also termed *neighborhood* of i. In this paper we focus on neighborhoods of agents defined by a metric interaction rule as follows:

$$\mathcal{N}_i(t) = \{j \in \mathcal{V};\ \|x_i(t) - x_j(t)\| \leq R\} \tag{2}$$

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where $\|.\|$ denote the usual Euclidean norm[1] and R is the radius within which agents are able to communicate. Let us remark that equation (2) indicates that the communication network given by metric interactions is symmetric (i.e. if agent i receives information from agent j, i also sends information to j). Metric interactions are usually assumed to be a good representation of how collective behavior takes place. Thus, most of the literature on the subject, including [12], [14], uses them.

We say that the agents achieve a *flocking behavior* if all the agents asymptotically move toward a common direction: there exists $v^* \in \mathbb{R}^d$ such that

$$\forall i \in \mathcal{V},\ \lim_{t\to+\infty} v_i(t) = v^*.$$

The goal of this paper is to determine easily checkable conditions (on the initial positions and velocities of agents) ensuring that a flocking behavior is achieved.

III. Preliminaries

In this section, we review some results from algebraic graph theory and multi-agent systems that will be useful in the subsequent discussion.

A. Algebraic Graph Theory

Let us recall some standard results from algebraic graph theory. More details can be found, for instance, in [6], [8].

An undirected graph G is a couple $(\mathcal{V},\mathcal{E})$ consisting of a set of nodes $\mathcal{V} = \{1,\dots,n\}$ and a set of edges given by a relation $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ that is symmetric ($(i,j)\in\mathcal{E}$ iff $(j,i)\in\mathcal{E}$) and anti-reflexive ($\forall i\in\mathcal{V}, (i,i)\notin\mathcal{E}$). If $(i,j)\in\mathcal{E}$, we say that i is a neighbor of j. The degree d_i of a node $i\in V$ is the number of neighbors of i. A path between i and j is a sequence of nodes $(i_1,i_2,\dots,i_p)$ such that $i_1 = i$, $i_p = j$ and $\forall k\in\{1,\dots,p-1\}, (i_k,i_{k+1})\in\mathcal{E}$. We shall consider throughout this paper paths without loops: for all $k,k'\in\{1,\dots,p-1\}$, $k\neq k'$ implies $i_k\neq i_{k'}$. A graph is said to be connected if for, every couple of nodes $(i,j)\in\mathcal{V}\times\mathcal{V}$ such that $i\neq j$, there exists a path between i and j. A graph $G' = (\mathcal{V},\mathcal{E}')$ is said to be a (spanning) subgraph of G if $\mathcal{E}'\subseteq\mathcal{E}$.

The Laplacian matrix $L = (l_{ij})$ of G is the $n\times n$ matrix defined for $i,j\in\mathcal{V}$ by

$$l_{ij} = \begin{cases} d_i & \text{if } i=j,\\ -1 & \text{if } (i,j)\in\mathcal{E},\\ 0 & \text{otherwise.}\end{cases} \tag{3}$$

This matrix is symmetric and positive semi-definite. Its eigenvalues are therefore nonnegative reals. 0 is an eigenvalue of L with eigenvector $\mathbb{1}_n = (1,\dots,1)$. We denote the eigenvalues of L by

$$0 = \lambda_1(G) \le \lambda_2(G) \le \ldots \le \lambda_n(G).$$

The second smallest eigenvalue of L, $\lambda_2(G)$ is usually referred to as the algebraic connectivity of graph G. If G is connected then $\lambda_2(G) > 0$. If G' is a subgraph of G then the second eigenvalue $\lambda_2(G')$ of its Laplacian matrix L' satisfies $\lambda_2(G')\le\lambda_2(G)$.

[1]In the following, $\|.\|$ will denote the usual Euclidean norm on $\mathbb{R}^d$ or $\mathbb{R}^{nd}$ depending on the context.

B. Consensus over Dynamic Networks

In this section, we recall a result from [11] which will be useful for our study.

Let $x(t) = (x_1(t),\dots,x_n(t)) \in \mathbb{R}^{nd}$ and $v(t) = (v_1(t),\dots,v_n(t))\in\mathbb{R}^{nd}$ be the stacked vectors of positions and velocities, respectively. We also define the stacked vectors of initial positions and velocities: $x^0 = (x_1^0,\dots,x_n^0)\in\mathbb{R}^{nd}$ and $v^0 = (v_1^0,\dots,v_n^0)\in\mathbb{R}^{nd}$. Let $\mathbf{L}(t) = L(t)\otimes I_d$ where $L(t)$ is the Laplacian matrix of graph $G(t)$, I_d is the $d\times d$ identity matrix and $\otimes$ denotes the Kronecker product. Then, the second equation in (1) becomes in matrix form

$$\dot v(t) = -\mathbf{L}(t)v(t). \tag{4}$$

Let $v^* = (v_1^0+\cdots+v_n^0)/n$ be the average value of the initial velocities. Since $L(t)$ is symmetric and $\mathbb{1}_n$ $\mathbb{1}$ is an eigenvector associated to eigenvalue 0, the average of the velocities is preserved by (4). It follows that if the agents achieve a flocking behavior, the common asymptotic velocity is necessarily v^*.

For $i\in\mathcal{V}$, let $\delta_i(t) = v_i(t) - v^*$, we define the *velocity disagreement vector* $\delta(t) = (\delta_1(t),\dots,\delta_n(t))$. Let $y_i(t) = x_i(t) - v^*t$ and $y(t) = (y_1(t),\dots,y_n(t))$, then $\dot y(t) = \delta(t)$. The vector $y(t)$ essentially gives the relative positions of the agents, as we have

$$\forall i,j\in\mathcal{V},\ x_i(t)-x_j(t) = y_i(t)-y_j(t).$$

Let us assume that the graph $G(t)$ remains in a set Π of connected graphs for all time $t\in\mathbb{R}^+$. The set of connected graphs with n nodes (and thus Π) is finite and therefore $\min_{G\in\Pi}\lambda_2(G)$ is well defined and strictly positive. We now state the following result from [11] which shows that if the graph $G(t)$ remains connected then the agents achieve a flocking behavior.

Theorem 1: [11] Let $I\subseteq\mathbb{R}^+$ be an interval containing 0, let $v: I\to\mathbb{R}^{nd}$ be a solution of (4). If graph $G(t)$ remains in a set Π of connected graphs for all $t\in I$, then for all $t\in I$

$$\|\delta(t)\| \le e^{-\kappa t}\|\delta(0)\|$$

where $\kappa = \min_{G\in\Pi}\lambda_2(G)$.

In [11], it is assumed that the agents evolve in a one-dimensional space ($d=1$). However, it is straightforward to extend this result to higher dimensions.

The previous result gives sufficient conditions to achieve flocking behavior. However, for arbitrary initial positions and velocities, it is unclear whether the graph $G(t)$ will remain connected for all $t\in\mathbb{R}^+$. In the following, we identify such a set of initial positions and velocities based on a measure of graph robustness.

IV. Graph Robustness Analysis

In this section, we define a notion of robustness for graphs defined using the metric interaction rule (2). For $i \in \mathcal{V}$, let $x_i \in \mathbb{R}^d$ be a position of agent i. Let $x = (x_1, \ldots, x_n)$, we define the associated graph $G_x = (\mathcal{V}, \mathcal{E}_x)$ with

$$\mathcal{E}_x = \{(i,j) \in \mathcal{V} \times \mathcal{V};\ \|x_i - x_j\| \leq R\}.$$

Let $x^0 \in \mathbb{R}^{nd}$ be a reference configuration for the positions of the agents. Assuming that G_{x^0} is a connected graph, we are interested in characterizing a neighborhood of x^0 such that for any perturbed configuration y in this neighborhood, the graph G_y is connected, though not necessarily equal to G_{x^0}. We introduce a measure of robustness for the graph G_{x^0} which allows us to identify such a neighborhood. We also show that there exists a connected subgraph of G_{x^0} which is also a subgraph of G_y for all y in this neighborhood. Finally, we provide an algorithm to compute this measure of robustness.

A. Measure of Robustness

Our measure of robustness of the graph G_{x^0} relies on the extend to which two agents can move away from each other before the communication is lost. This is measured by the *slackening* of a path $(i_1, i_2, \ldots, i_p)$ of G_{x^0} defined by

$$s(i_1, i_2, \ldots, i_p) = \min_{k=1}^{p-1} (R - \|x^0_{i_k} - x^0_{i_{k+1}}\|).$$

By definition of G_{x^0}, we have that, for all paths $(i_1, i_2, \ldots, i_p)$, $0 \leq s(i_1, i_2, \ldots, i_p) \leq R$. Intuitively, if the distances between agents do not change more than $s(i_1, i_2, \ldots, i_p)$, then the path $(i_1, i_2, \ldots, i_p)$ is preserved.

We can now define the *path-robustness* ρ_{ij} between two agents i and $j \in \mathcal{V}$ with $i \neq j$, as the maximal slackening of all paths between i and j:

$$\rho_{ij} = \max_{(i_1, i_2, \ldots, i_p) \in \text{Paths}(i,j)} s(i_1, i_2, \ldots, i_p)$$

where $\text{Paths}(i,j)$ is the set of all paths from i to j in G_{x^0}. Since G_{x^0} is assumed to be connected, $\text{Paths}(i,j)$ is not empty and for all $i, j \in \mathcal{V}$, $0 \leq \rho_{ij} \leq R$. Also, for $i \in \mathcal{V}$, we set $\rho_{ii} = R$. Intuitively, if the distances between agents do not change more than ρ_{ij} then there remains at least one path between agents i and j.

Finally, the *robustness* $\rho_{G_{x^0}}$ of the graph G_{x^0} is defined as the minimal path-robustness between all pairs of nodes

$$\rho_{G_{x^0}} = \min_{(i,j) \in \mathcal{V}^2} \rho_{ij}.$$

Then, we have $0 \leq \rho_{G_{x^0}} \leq R$. If the distances between agents do not change more than $\rho_{G_{x^0}}$ then for any two agents $i, j \in \mathcal{V}$, there remains a path between i and j and therefore the graph remains connected. This will be proved formally in Proposition 4. Moreover, we will also show that a subgraph of G_{x^0} (named core robust subgraph) is preserved.

Definition 2: The *core robust subgraph* of G_{x^0} is the graph $\mathcal{M}(G_{x^0}) = (\mathcal{V}, \mathcal{M}(\mathcal{E}_{x^0}))$ where

$$\mathcal{M}(\mathcal{E}_{x^0}) = \{(i,j) \in \mathcal{V} \times \mathcal{V};\ \|x^0_i - x^0_j\| \leq R - \rho_{G_{x^0}}\}.$$

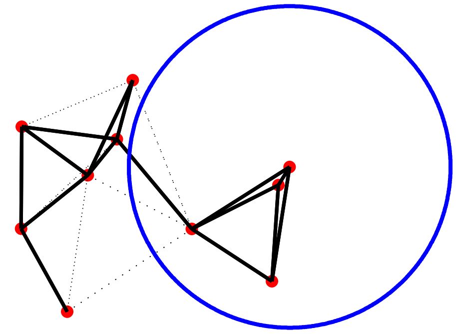

Fig. 1. Graph G_{x^0} (all lines) and the associated core robust subgraph $\mathcal{M}(G_{x^0})$ (thicker lines only). The circle represents the maximum communication radius of the top right agent

Let us remark that since $\rho_{G_{x^0}} \geq 0$, $\mathcal{M}(G_{x^0})$ is clearly a subgraph of G_{x^0}, the following result states that it is connected:

Lemma 3: Let $x^0 \in \mathbb{R}^{nd}$ be a reference configuration such that the associated graph G_{x^0} is connected. Then, the core robust subgraph $\mathcal{M}(G_{x^0})$ is connected.

Proof: Let i and $j \in \mathcal{V}$, then $\rho_{ij} \geq \rho_{G_{x^0}}$. Let $(i_1, i_2, \ldots, i_p)$ be a path of G_{x^0} between i and j with maximal slackening. Hence, $s(i_1, i_2, \ldots, i_p) = \rho_{ij}$. Then, for all $k \in \{1, \ldots, p-1\}$,

$$R - \|x^0_{i_k} - x^0_{i_{k+1}}\| \geq s(i_1, i_2, \ldots, i_p) \geq \rho_{G_{x^0}}.$$

Therefore, $(i_k, i_{k+1}) \in \mathcal{M}(\mathcal{E}_{x^0})$, for all $k \in \{1, \ldots, p-1\}$. It follows that $(i_1, i_2, \ldots, i_p)$ is a path of $\mathcal{M}(G_{x^0})$ between i and j. Thus, $\mathcal{M}(G_{x^0})$ is connected. ■

An example of a core robust subgraph is shown in Figure 1. We can now state the main result of this section.

Proposition 4: Let $x^0 \in \mathbb{R}^{nd}$ be a reference configuration such that the associated graph G_{x^0} is connected. Let $y \in \mathbb{R}^{nd}$ be a perturbed configuration such that

$$\|y - x^0\| \leq \frac{\rho_{G_{x^0}}}{\sqrt{2}}. \quad (5)$$

Then, $\mathcal{M}(G_{x^0})$ is a subgraph of G_y and G_y is connected.

Proof: Let $z = (z_1, \ldots, z_n)$ such that $z = y - x^0$. For all $i, j \in \mathcal{V}$, $-2z_i \cdot z_j \leq \|z_i\|^2 + \|z_j\|^2$. Then, it follows that

$$\begin{aligned} \|z_i - z_j\|^2 &= \|z_i\|^2 + \|z_j\|^2 - 2z_i \cdot z_j \\ &\leq 2(\|z_i\|^2 + \|z_j\|^2) \\ &\leq 2\|z\|^2 = 2\|y - x^0\|^2. \end{aligned}$$

Then, from (5), we have for all $i, j \in \mathcal{V}$, $\|z_i - z_j\| \leq \rho_{G_{x^0}}$. Let $(i,j) \in \mathcal{M}(\mathcal{E}_{x^0})$, then by Definition 2, it follows that $\|x^0_i - x^0_j\| \leq R - \rho_{G_{x^0}}$. Thus,

$$\begin{aligned} \|y_i - y_j\| &= \|x^0_i - x^0_j + z_i - z_j\| \\ &\leq \|x^0_i - x^0_j\| + \|z_i - z_j\| \\ &\leq R - \rho_{G_{x^0}} + \rho_{G_{x^0}} \\ &\leq R. \end{aligned}$$

This means that $(i,j) \in \mathcal{E}_y$. Therefore $\mathcal{M}(G_{x^0})$ is a subgraph of G_y. By Lemma 3, $\mathcal{M}(G_{x^0})$ is connected. Therefore G_y is connected as well. ■

B. Computation of the Graph Robustness

In the previous paragraph, we characterized a neighborhood of the reference configuration such that the connectivity of the interaction graph is preserved. For Proposition 4 to be useful, the robustness $\rho_{G_{x^0}}$ must be computable. This can be done using dynamic programming by adapting the Floyd-Warshall algorithm for computing shortest paths in graphs (see e.g. [2]) as shown in Algorithm 1. It runs with space-complexity $O(n^2)$ and time-complexity $O(n^3)$.

Algorithm 1 Computation of the robustness $\rho_{G_{x^0}}$

```
// Initialization:
∀(i,j) ∈ V², ρ⁰_ij ← R − ‖x⁰_i − x⁰_j‖;
// Main loop:
for k ∈ V do
  for i ∈ V do
    for j ∈ V do
      ρ^k_ij ← max(ρ^{k−1}_ij, min(ρ^{k−1}_ik, ρ^{k−1}_kj));
    end for
  end for
end for
// Computation of robustness:
ρ_{G_x0} = min_{(i,j)∈V²} ρ^n_ij;
```

The main idea of Algorithm 1 is as follows. Let ρ_{ij}^k denote the maximal slackening of all paths $(i_1, i_2, \ldots, i_p)$ between i and j and with intermediate nodes $i_2, \ldots, i_{p-1}$ in $\{1, \ldots, k\}$. If there does not exist a path between i and j with intermediate nodes in $\{1, \ldots, k\}$, let $\rho_{ij}^k < 0$. Then, to compute ρ_{ij}^k from ρ_{ij}^{k-1}, either the maximal slackening path between i and j with intermediates nodes in $\{1, \ldots, k\}$ does not contain node k and $\rho_{ij}^k = \rho_{ij}^{k-1}$ or it contains node k and $\rho_{ij}^k = \min(\rho_{ik}^{k-1}, \rho_{kj}^{k-1})$. Finally, the path-robustness $\rho_{ij} = \rho_{ij}^n$.

The time and space complexity of Algorithm 1 are in $O(n^3)$ and $O(n^2)$, respectively. Let us remark that since the graph G_{x^0} is symmetric, it is clear that for all $i, j, k \in \mathcal{V}$, we have $\rho_{ij}^k = \rho_{ji}^k$. Then, in Algorithm 1, it is actually only necessary to compute ρ_{ij}^k for $j \leq i$, thus dividing time complexity by two.

V. Sufficient Conditions for Flocking

We now have the necessary ingredients, that is Theorem 1 and Proposition 4, to establish sufficient conditions for flocking. Let us rewrite equation (1) in matrix form:

$$\begin{aligned} \dot{x}(t) &= v(t) \\ \dot{v}(t) &= -\mathbf{L}(t)v(t). \end{aligned} \tag{6}$$

Let $y(t)$ and $\delta(t)$ be defined as in section III-B. $\delta(t)$ is the disagreement vector: a flocking behavior is achieved if and only if $\delta(t)$ asymptotically goes to 0.

A. Main Result

The following theorem gives sufficient conditions on initial positions and velocities under which a flocking behavior is achieved. It relates the initial value of the disagreement vector, the robustness of the initial graph G_{x^0} and the algebraic connectivity of the core robust subgraph $\mathcal{M}(G_{x^0})$. Moreover, the theorem states that under these conditions, the communication network remains connected for all time:

Theorem 5: Let $x^0 \in \mathbb{R}^{nd}$ be a vector of initial positions of the agents such that the associated graph G_{x^0} is connected and its robustness $\rho_{G_{x^0}} > 0$. Let $v^0 \in \mathbb{R}^{nd}$ be a vector of initial velocities such that its corresponding disagreement vector $\delta(0)$ verifies

$$\|\delta(0)\| \leq \frac{\lambda_2^* \rho_{G_{x^0}}}{\sqrt{2}} \tag{7}$$

where $\lambda_2^* = \lambda_2(\mathcal{M}(G_{x^0}))$. Then, for all $t \in \mathbb{R}^+$, $\mathcal{M}(G_{x^0})$ is a subgraph of $G(t)$. Moreover,

$$\|y(t) - x^0\| \leq \frac{\|\delta(0)\|}{\lambda_2^*}$$

and

$$\|\delta(t)\| \leq e^{-\lambda_2^* t}\|\delta(0)\|.$$

Proof: Let Π be the set of graphs with n nodes which have $\mathcal{M}(G_{x^0})$ as a subgraph. Since G_{x^0} is connected, we have by Lemma 3, that $\mathcal{M}(G_{x^0})$ is connected. Therefore, all graphs in Π are connected and since $\mathcal{M}(G_{x^0}) \in \Pi$,

$$\min_{G \in \Pi} \lambda_2(G) = \lambda_2(\mathcal{M}(G_{x^0})) = \lambda_2^* > 0.$$

Let us assume that there exists $t > 0$ such that $\mathcal{M}(G_{x^0})$ is not a subgraph of $G(t)$ (i.e. $G(t) \notin \Pi$). Let

$$t^* = \inf\{t \in \mathbb{R}^+;\ G(t) \notin \Pi\}.$$

If $t^* > 0$, it follows from Theorem 1 that for all $t \in [0, t^*)$

$$\|\delta(t)\| \leq e^{-\lambda_2^* t}\|\delta(0)\| \leq e^{-\lambda_2^* t}\frac{\lambda_2^* \rho_{G_{x^0}}}{\sqrt{2}}.$$

By remarking that

$$y(t) = x^0 + \int_0^t \delta(s)ds$$

we have for all $t \in [0, t^*)$

$$\|y(t) - x^0\| \leq \frac{\lambda_2^* \rho_{G_{x^0}}}{\sqrt{2}} \int_0^t e^{-\lambda_2^* s}ds < \frac{\rho_{G_{x^0}}}{\sqrt{2}}.$$

Then, by continuity of y, there exists $\varepsilon > 0$ such that for all $t \in [0, t^* + \varepsilon]$,

$$\|y(t) - x^0\| \leq \frac{\rho_{G_{x^0}}}{\sqrt{2}}.$$

If $t^* = 0$, since $y(0) = x^0$ and by continuity of y, the same kind of property holds. By Proposition 4, we have for all $t \in [0, t^* + \varepsilon], G_{y(t)} \in \Pi$. By remarking that for all $i, j \in \mathcal{V}, x_i(t) - x_j(t) = y_i(t) - y_j(t)$, it follows that the graph $G(t) = G_{x(t)} = G_{y(t)}$. Thus, for all $t \in [0, t^* + \varepsilon], G(t) \in \Pi$. This contradicts the definition of t^*. Therefore, for all $t \in \mathbb{R}^+, G(t) \in \Pi$. This proves the first part of the theorem. Then, from Theorem 1, it follows that for all $t \in \mathbb{R}^+$,

$$\|\delta(t)\| \leq e^{-\lambda_2^* t}\|\delta(0)\|$$

and

$$\|y(t) - x^0\| \leq \int_0^t \|\delta(s)\| ds \leq \int_0^t e^{-\lambda_2^* s} \|\delta(0)\| ds \leq \frac{\|\delta(0)\|}{\lambda_2^*}.$$

■

The upper bound on the initial disagreement vector given by Theorem 5 is proportional to both the robustness of the graph and the algebraic connectivity of its core robust subgraph. In the case of graphs based on a metric interaction rule, these two quantities tend to increase with the density of the agents. Therefore, Theorem 5 shows that the higher the disagreement between initial velocities, the denser the group of agents must initially be to ensure that a flocking behavior will be achieved. This observation is reasonable. Notice that the robustness and algebraic connectivity of a graph are bounded by R and n respectively. This shows that when the initial velocity disagreement is too high then there is not always a configuration of agents ensuring than a flocking behavior will be achieved.

Theorem 5 gives sufficient conditions for flocking. An important question is whether the bound on the initial disagreement vector is optimal: i.e. can we find a configuration of agents where the non-respect of the inequality (7) prevents the agents from achieving a flocking behavior. Such a configuration is presented in the next paragraph.

B. Tightness of the bound

We consider a set of two agents evolving in the one-dimensional space $\mathbb{R}$. Initially, the agents have the same position $x_1^0 = x_2^0 = 0$. This implies that G_{x^0} is the complete graph of order 2. The robustness of the graph is maximal $\rho_{G_{x^0}} = R$ and $\mathcal{M}(G_{x^0}) = G_{x^0}$. It follows that $\lambda_2^* = 2$.

Initially, the agents move with opposite velocities $v_1^0 = -v_2^0 = -\alpha$, for some $\alpha > 0$. Then, the norm of the disagreement vector is $\|\delta(0)\| = \|v^0\| = \sqrt{2}\alpha$. We choose α such that equation (7) is not satisfied, that is $\alpha > R$. We now show that the distance between the two agents eventually becomes greater than R and that after this time, the two agents move independently in opposite directions.

While $\|x_2(t) - x_1(t)\| \leq R$, the communication network remains the same. From equation (1), we have

$$\dot{v}_2(t) - \dot{v}_1(t) = -2(v_2(t) - v_1(t))$$

. Since $v_2(0) - v_1(0) = 2\alpha$ and $x_2(0) - x_1(0) = 0$,

$$v_2(t) - v_1(t) = 2\alpha e^{-2t}$$

and

$$x_2(t) - x_1(t) = \alpha(1 - e^{-2t}).$$

At time $T = -\frac{1}{2}\log(1 - R/\alpha)$, $\|x_2(T) - x_1(T)\| = R$ and the communication link is broken. Then, equation (1) gives $\dot{v}_1(t) = \dot{v}_2(t) = 0$. Then, for all $t \geq T$,

$$v_2(t) - v_1(t) = 2(\alpha - R) > 0$$

and

$$x_2(t) - x_1(t) = R + 2(\alpha - R)(t - T).$$

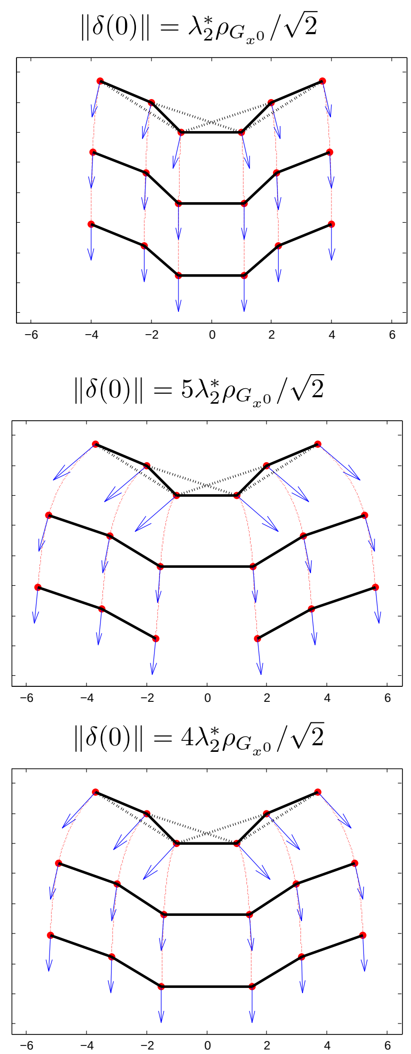

Fig. 2. Evolution of a group of mobile agents in a conflicting situation for several values of the initial disagreement vector norm. Lines represent communication links. Thicker lines belong to the core robust subgraph $\mathcal{M}(G_{x^0})$ whereas dashed lines do not. The state of the system is shown at times $t = 0$, $t = 60$ and $t = 120$. The trajectories of agents are displayed in thin dotted lines. Initially, agents divide in two subgroups holding velocities with opposite x-coordinates (blue arrows). Communication radius is $R = 3.2$. Velocity arrows were increased 3 times for clarity.

Thus, after time T, the two agents become independent and continue to evolve at a constant velocity in opposite directions. The flocking behavior is not achieved.

VI. Simulations

In this section, we present simulations illustrating our approach. Several facts are underlined. Firstly, when the bound on the norm of the initial disagreement vector given by equation (7) is respected, then we show that communication links belonging to the core robust subgraph are preserved while others can be broken. Secondly, we show that the conditions in Theorem 5 are only sufficient and not necessary by showing that the connectivity can be preserved and the flocking behavior achieved even though the norm of the initial disagreement vector does not satisfy equation (7).

To make the simulations more concrete, one can imagine the following conflicting situation within a group of moving agents. Initially, agents divide in two halves with velocities bearing opposite x-coordinates. The initial positions induces a connected communication network. We wonder whether the group will eventually stay together and achieve a flocking behavior or rather split apart because of the initial conflict. As one could expect, simulations show that the answer depends on the strength of the initial velocities of the agents.

In the first simulation, at the top of Figure 2, the bound on the norm of the initial disagreement vector given by equation (7) is respected. The agents start moving away from each other but their difference of velocities rapidly vanishes and converges to 0 due to the interactions between agents. Thus, the interaction graph remains connected for ever. Also, let us remark that links which did not belong to the initial core robust subgraph $\mathcal{M}(G_{x^0})$ were not preserved. This emphasizes the fact that Theorem 5 only guarantees the conservation of the communication links inside the initial core robust subgraph $\mathcal{M}(G_{x^0})$.

In the second simulation, in the middle of Figure 2, the agents start with higher difference of velocities so that the norm of the disagreement vector is 5 times larger than what it should be to satisfy equation (7). The difference of velocities between the two central agents starts vanishing due to their communication link. Before the dynamics had time to even out all conflicting velocities, the central agents moved away from each other to a distance larger than the interaction radius, and thus their communication link was broken. The group divided without converging toward a common velocity and its two subgroups separated indefinitely. This is a possible outcome when the sufficient condition given in Theorem 5 is not satisfied.

The last simulation, at the bottom of Figure 2, shows an intermediate case between the first two simulations. Even though the norm of the initial disagreement vector does not satisfy equation (7), the agents still achieve a flocking behavior. This stresses that our approach is conservative and that the given condition is sufficient but not necessary.

VII. Conclusion

In this paper, we have considered a multi-agent system consisting of mobile agents with second-order dynamics and where the communication network is determined by a metric interaction rule. Our approach builds upon earlier results of Olfati-Saber and Murray [11] and links algebraic connectivity of the communication network to the speed of convergence towards consensus. We have established sufficient conditions on the initial positions and velocities of the agents which ensure that the agents will asymptotically achieve flocking. Our main contribution has been to propose a suitable notion of robustness of graphs induced by the metric interaction rule. The robustness of a connected graph can be understood as the level of disturbance on the distances between agents that can be accommodated without disconnecting the graph. Our main result states that whenever the initial velocity disagreement among agents is smaller than a threshold (formed with the robustness and the algebraic connectivity of the graph), the agents will achieve a flocking behavior. The main interest of this approach is the possibility of ensuring flocking a priori. The sufficient condition can be easily verified through rapid computation.

We already see two possible extensions of this work. The first one consists in refining our sufficient condition. As pointed out in the simulation section, the condition given by equation (7) is fairly conservative. This is in part due to the fact that the disagreement measure only takes into account the velocities, it would be more informative to relate velocity with position because two agents with opposite velocities have more chance to agree on their velocity if they point toward each other, than if they point away from each other. Also, a subgroup of agents with high connectivity is intuitively more inclined to agree on their velocity than a subgroup of low connectivity. Thus, agents belonging to a highly connected local neighborhood should be allowed higher initial velocities. The second set of extensions consists in the adaptation of the present method to different systems. For instance, it should be possible to include time-delays in the framework. Similarly, one can include uncertainty in the model by adding motion disturbance or stochasticity in interactions between agents.

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