Motion Planning for Nonlinear Systems using Hybridizations and Robust Controllers on Simplices

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Abstract— In this paper, we consider a motion planning problem for a class of constrained nonlinear systems. In each simplex of a triangulation of the set of states, the nonlinear dynamics is conservatively approximated by an affine system subject to disturbances. This results in a hybrid abstraction, called *hybridization*, of the nonlinear control system. Except for the disturbance, this hybridization can be seen as a piecewise affine hybrid system on simplices for which motion planning techniques have been developed by Habets and van Schuppen in a series of papers. We extend these techniques to handle the disturbances by synthesizing robust affine controllers on the simplices of the triangulation. Our approach, though conservative, can be fully automated and is computationally tractable. We illustrate our method on an example.

I. INTRODUCTION

Motion planning is an important problem in robotics where it has attracted a lot of attention (see [17] for an extensive review of the subject). Until recently, the work on this problem could be roughly classified into two main approaches. The first approach assumed unconstrained dynamics of the robots and focused on the complexity of the environment while the second used a detailed dynamic model of the robot and assumed an unconstrained state-space.

In the past decade, several approaches have been proposed for handling in the same framework the complexity of both the dynamics and the environment. Starting from a path obtained from unconstrained dynamics, some methods modify these paths in order to make them satisfy the differential constraints [10], [16]. Several sampling based techniques have also been proposed for exploring the set of possible trajectories of the robot [7], [18]. Building on earlier work by Habets and van Schuppen [12], [13], [11], symbolic control approaches [3] compute a discrete partition of the state-space, design an abstract path at the discrete level and refine it using local continuous controllers in each domain of the partition [5], [9], [15]. This latter approach has been limited to systems with affine or multi-affine dynamics for which the algorithmic synthesis of local controllers is possible [11], [20], [4].

In this paper, we extend this approach to handle systems with nonlinear dynamics. In each simplex of a triangulation of the set of states, the nonlinear dynamics is conservatively approximated by an affine system subject to disturbances.

This results in a hybrid abstraction, called *hybridization* [2], of the nonlinear control system. Except for the disturbance, this hybridization can be seen as a piecewise affine hybrid system on simplices for which motion planning techniques have been developed in [11]. We extend these techniques to handle the disturbances by developping local robust controllers on the simplices of the triangulation. Our approach, though conservative, can be fully automated and is computationally tractable.

The paper is organized as follows. In the next section, we formulate the motion planning problem we consider. Then, we introduce the notion of hybridization and show how our nonlinear motion planning problem can be replaced by a robust hybrid motion planning problem. Then, we extend the techniques of [11], [20] for computing local robust controllers on simplices and use these controllers to solve the hybrid motion planning problem. Finally, we illustrate the method on an example.

II. PROBLEM FORMULATION

In this paper, we consider a constrained nonlinear system of the following form:

$$\Sigma : \begin{cases} \dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) + g(\mathbf{x}(t))\mathbf{u}(t), \\ \mathbf{x}(t) \in D, \ \mathbf{u}(t) \in U. \end{cases} \quad (1)$$

The *state domain* $D \subseteq \mathbb{R}^n$ is assumed to be a compact polytope, possibly non-convex and with holes. Equivalently, D can be seen as a finite union of convex compact polytopes. The *control set* $U \subseteq \mathbb{R}^p$ is assumed to be a convex compact polytope. Additionnally, we shall assume that the maps $f : D \rightarrow \mathbb{R}^n$ and $g : D \rightarrow \mathbb{R}^{n \times p}$ are of class $\mathcal{C}^2$ and $\mathcal{C}^1$ respectively. A continuous and piecewise $\mathcal{C}^1$ map $\mathbf{x} : \mathbb{R}^+ \rightarrow D$ is a *trajectory* of Σ under the piecewise continuous control input $\mathbf{u} : \mathbb{R}^+ \rightarrow U$ if equation (1) holds for all $t \in \mathbb{R}^+$ where $\mathbf{x}$ is differentiable. We consider the following motion planning problem for system Σ:

Problem 2.1 (Motion planning): Let $I \subseteq D$ and $F \subseteq D$ be a set of initial states and a set of final states, respectively. Design a controller $h : D' \rightarrow U$, where $I \subseteq D' \subseteq D$, such that for all $x_0 \in I$, the trajectory of system Σ, $\mathbf{x} : \mathbb{R}^+ \rightarrow D$, with $\mathbf{x}(0) = x_0$ and given by

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) + g(\mathbf{x}(t))h(\mathbf{x}(t)), \quad (2)$$

satifies the following properties:

- For all $t \in \mathbb{R}^+$, $\mathbf{x}(t) \in D' \subseteq D$;
- There exists $T \in \mathbb{R}^+$, such that for all $t \geq T$, $\mathbf{x}(t) \in F$.

We assume that the sets I and F are given by possibly non-convex compact polytopes. Essentially, the goal of the

motion planning problem is to design a state-feedback controller (not necessarily defined on the entire state domain D) and such that all the trajectories of Σ with an initial state in I remain in the domain D and eventually reach the set F and stay there forever. Let us remark that the controller h need not be continuous.

Our approach to problem 2.1 consists of two main ingredients: a hybridization of the nonlinear system Σ and robust controllers on simplices.

III. HYBRIDIZATION

A *hybridization* is a hybrid abstraction of a continuous dynamical system (see *e.g.* [2]). Conservativeness of the approximation is ensured by the introduction of disturbances. Hybridizations have been used in the context of reachability analysis of nonlinear continuous systems in [14], [1], [2]. In this paper, we shall use such abstractions for solving the motion planning problem 2.1. In the following, $\|.\|$ denotes the Euclidean norm (for vectors) and the associated induced norm (for matrices).

A. Principle

We describe the main features of hybridizations. The details about their computation will be given in the following paragraph.

Definition 3.1: $\mathcal{S} = \{S_1, \ldots, S_N\}$ is a *triangulation* of the domain D if:

- For all $S_i \in \mathcal{S}$, S_i is a full dimensional simplex[1] of $\mathbb{R}^n$;
- For all $S_i, S_j \in \mathcal{S}$, their intersection is either the convex hull of their common vertices or empty;
- $S_1 \cup \cdots \cup S_N = D$.

Let us further assume that $\mathcal{S}$ contains a triangulation of I, $\mathcal{S}_I \subseteq \mathcal{S}$, and a triangulation of F, $\mathcal{S}_F \subseteq \mathcal{S}$. We now consider the following piecewise affine hybrid system with bounded disturbances defined over the triangulation $\mathcal{S}$:

$$\Sigma' : \begin{cases} \dot{\mathbf{x}}(t) = A_i \mathbf{x}(t) + B_i \mathbf{u}(t) + a_i + \mathbf{d}(t), \\ \mathbf{u}(t) \in U, \|\mathbf{d}(t)\| \leq \mu_i, \text{ if } \mathbf{x}(t) \in S_i. \end{cases} \quad (3)$$

A continuous and piecewise $\mathcal{C}^1$ map $\mathbf{x} : \mathbb{R}^+ \to D$ is a trajectory of Σ' under the piecewise continuous control input $\mathbf{u} : \mathbb{R}^+ \to U$ if there exists a piecewise continuous disturbance $\mathbf{d} : \mathbb{R}^+ \to \mathbb{R}^n$ such that equation (3) holds for all $t \in \mathbb{R}^+$ where $\mathbf{x}$ is differentiable. Let us remark that, with the exception of the disturbance, the system Σ' belongs to the class of piecewise affine hybrid systems on simplices for which motion planning techniques have been developed in [11].

Definition 3.2: We say that the piecewise affine hybrid system Σ' is a *hybridization* of the nonlinear control system Σ if for all $i \in \{1, \ldots, N\}$,

$$\max_{x \in S_i, u \in U} \|f(x) + g(x)u - A_i x - B_i u - a_i\| \leq \mu_i. \quad (4)$$

The following result shows that the hybridization Σ' is a conservative approximation of system Σ.

[1] A simplex is the convex hull of $n + 1$ affinely independent points.

Proposition 3.3: Let $\mathbf{x}$ be the trajectory of nonlinear system Σ under the control input $\mathbf{u}$ and with initial state x_0. Then, $\mathbf{x}$ is a trajectory of the hybridization Σ' under the control input $\mathbf{u}$ and with initial state x_0.

Proof: It is sufficient to prove that there exists a disturbance $\mathbf{d}$ such that equation (3) holds. Let us consider the disturbance given by, when $\mathbf{x}(t) \in S_i$:

$$\mathbf{d}(t) = f(\mathbf{x}(t)) + g(\mathbf{x}(t))\mathbf{u}(t) - A_i \mathbf{x}(t) - B_i \mathbf{u}(t) - a_i.$$

From equation (4), it follows that $\|\mathbf{d}(t)\| \leq \mu_i$ whenever $\mathbf{x}(t) \in S_i$. Then, it is straightforward to check that equation (3) holds. ∎

We now define a motion planning problem for the piecewise affine hybrid system Σ':

Problem 3.4 (Robust hybrid motion planning): Design a controller $h : D' \to U$ where $I \subseteq D' \subseteq D$ such that for all $x_0 \in I$, for all the trajectories of system Σ', $\mathbf{x} : \mathbb{R}^+ \to D$, with $\mathbf{x}(0) = x_0$ and given by, when $\mathbf{x}(t) \in S_i$,

$$\dot{\mathbf{x}}(t) = A_i \mathbf{x}(t) + B_i h(\mathbf{x}(t)) + a_i + \mathbf{d}(t), \|\mathbf{d}(t)\| \leq \mu_i,$$

the following properties hold:

- For all $t \in \mathbb{R}^+$, $\mathbf{x}(t) \in D' \subseteq D$;
- There exists $T \in \mathbb{R}^+$, such that for all $t \geq T$, $\mathbf{x}(t) \in F$.

Let us remark that this is a robust control problem since the specified property must hold for all admissible disturbances $\mathbf{d}$. We have as an immediate consequence of Proposition 3.3:

Corollary 3.5: If a controller h solves problem 3.4, then it solves problem 2.1.

We now propose a method for the computation of a hybridization Σ' of a nonlinear system Σ.

B. Computation of the hybridization

We do not discuss the computation of a triangulation $\mathcal{S}$ of the domain D. This is a well studied problem in computational geometry for which efficient algorithms exist, at least in low dimensional spaces (see *e.g.* [19]). In higher dimensional spaces, and provided the domain can be partionned in hypercubes, a triangulation of D can be obtained using a simple triangulation of each hypercube as shown in [2]. It is to be noted that the size of the simplices of the triangulation can generally be made arbitrary small. We shall use the notation δ_i for the *diameter* of the simplex S_i defined by

$$\delta_i = \max_{x,x' \in S_i} \|x - x'\|.$$

We now focus on the computation of the affine dynamics in a simplex $S_i \in \mathcal{S}$. Let $\{v_i^0, \ldots, v_i^n\}$ denote the $n + 1$ vertices of the simplex. Essentially, the map $f(x)$ is approximated by the affine vector field $A_i x + a_i$ while the map $g(x)$ is approximated by the constant matrix B_i. Thus, A_i and a_i are chosen such that

$$\forall j \in \{0, \ldots, n\}, \ A_i v_i^j + a_i = f(v_i^j).$$

Since $v_i^0, \ldots, v_i^n$ are affinely independent, this condition uniquely determines A_i and a_i which can be computed by

solving a system of linear equations. Then, the matrix B_i is defined by

$$B_i = \frac{1}{n+1} \sum_{j=0}^{n} g(v_i^j)$$

It remains to determine the disturbance bound μ_i such that equation (4) holds. The following proposition gives an admissible value for μ_i:

Proposition 3.6: Let μ_i be given by:

$$\mu_i = \alpha_i \delta_i^2 + \beta_i \delta_i \max_{u \in U} \|u\|$$

where

$$\alpha_i = \frac{n^2 \sqrt{n}}{2(n+1)^2} \max_{l=1}^{n} \max_{x \in S_i} \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \left\| \frac{\partial^2 f_l(x)}{\partial x_{k_1} \partial x_{k_2}} \right\|$$

and

$$\beta_i = \frac{n}{n+1} \sqrt{ \sum_{k=1}^{n} \sum_{l=1}^{p} \sup_{x \in S_i} \|\nabla g_{k,l}(x)\|^2 }.$$

Then, equation (4) holds.

Proof: Using the triangular inequality, it is sufficient to show that for all $x \in S_i$,

$$\|f(x) - A_i x - a_i\| \leq \alpha_i \delta_i^2 \text{ and } \|g(x) - B_i\| \leq \beta_i \delta_i.$$

A detailed proof of the first inequality can be found in [2]. Bounding the matrix induced norm by the Frobenius norm and using the mean value theorem, we can show that for all $x, x' \in S_i$,

$$\|g(x) - g(x')\| \leq \|x - x'\| \sqrt{ \sum_{k=1}^{n} \sum_{l=1}^{p} \sup_{x \in S_i} \|\nabla g_{k,l}(x)\|^2 }.$$

Then, using techniques similar to those in [2], the second inequality follows. ∎

More than giving a procedure to compute a bound μ_i which may be actually easier to derive directly from equation (4), the previous proposition points out the fact that the bound on the disturbances in the hybridization can be made arbitrarily small provided the triangulation of the domain is sufficiently fine.

The rest of the paper is devoted to solving problem 3.4. Our approach is essentially an extension of the techniques developed in [11] for the class of piecewise affine hybrid systems on simplices to the class of systems with disturbances. It is based on the synthesis of local robust controllers on the simplices of the triangulation making it possible to drive all the trajectories within a simplex through a set of specified facets[2]. Then, global motion planning is ensured at the triangulation level.

IV. ROBUST CONTROLLERS ON SIMPLICES

The results presented in this section extend those of [11] and [20] to the class of affine systems with disturbances on simplices. The synthesis of local controllers requires some preliminary results on autonomous affine systems with disturbances.

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[2]A facet of the simplex S is the convex hull of n vertices of S.

A. Affine systems with disturbances on simplices

Let S be a simplex of $\mathbb{R}^n$, we consider the following autonomous affine system with disturbances :

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + a + \mathbf{d}(t), \ \mathbf{x}(t) \in \mathbb{R}^n, \ \|\mathbf{d}(t)\| \leq \mu \quad (5)$$

where $\mathbf{d} : \mathbb{R}^+ \to \mathbb{R}^n$ is a piecewise continuous disturbance. We denote $v_0, \dots, v_n$ and $F_0, \dots, F_n$ the vertices and the facets of S with the convention that F_j is the facet opposite to vertex v_j. $m_0, \dots, m_n$ denote the outward unit normal vectors of the facets of S.

We say that a trajectory $\mathbf{x}$ of (5), starting in S, *exits S at time $T \geq 0$*, if there exists $\varepsilon > 0$ such that

$$\forall t \in [0, T], \ \mathbf{x}(t) \in S \text{ and } \forall t \in (T, T+\varepsilon), \ \mathbf{x}(t) \notin S.$$

We first establish necessary and sufficient conditions such that all the trajectories of (5), starting in S, exit S in finite time.

Proposition 4.1: The following assertions are equivalent:
(i) All the trajectories of (5) starting in S, exits S in finite time.
(ii) There exists $\xi \in \mathbb{R}^n$, $\|\xi\| = 1$, such that for all $x \in S$, $\xi^T(Ax + a) > \mu$.
(iii) For all $x \in S$, $\|Ax + a\| > \mu$.

Proof: The proof is adapted from [20].

(i) $\implies$ (iii): If there exists $x \in S$, such that $\|Ax+a\| \leq \mu$, then there exists a constant trajectory of (5) given by $\mathbf{x}(t) = x$ which does not exit S.

(iii) $\implies$ (ii): Let $G = \{Ax+a|\ x \in S\}$, G is a convex set. Let $v_0 \in G$ be the projection of 0 on G. From the projection theorem (see *e.g.* [6], page 88), for all $v \in G$, $v_0^T v \geq \|v_0\|^2$. From (iii), $\|v_0\| > \mu$, let $\xi = v_0 / \|v_0\|$, then $\|\xi\| = 1$ and for all $v \in G$, $\xi^T v > \mu$.

(ii) $\implies$ (i): Assume (ii) holds but not (i), then there exists $\mathbf{x}$ a trajectory of (5) starting in S and staying in S forever. Let $\nu = \min\{\xi^T(Ax + a)|\ x \in S\} - \mu$, from (ii), $\nu > 0$. Then, for all $t \in \mathbb{R}^+$,

$$
\begin{aligned}
\xi^T \mathbf{x}(t) &= \xi^T \mathbf{x}(0) + \int_0^t \xi^T(A\mathbf{x}(s) + a + \mathbf{d}(s))ds \\
&\geq \xi^T \mathbf{x}(0) + \int_0^t (\xi^T(A\mathbf{x}(s) + a) - \mu)ds \\
&\geq \xi^T \mathbf{x}(0) + \int_0^t \nu ds = \xi^T \mathbf{x}(0) + \nu t
\end{aligned}
$$

which contradicts the fact that the trajectory $\mathbf{x}$ remains in the bounded set S. ∎

The previous property can be characterized using only the value of the affine vector field at the vertices of the simplex:

Theorem 4.2: All the trajectories of (5) starting in S, exits S in finite time if and only if

$$\text{Conv}(\{Av_0 + a, \dots, Av_n + a\}) \cap \mathbf{B}(0, \mu) = \emptyset. \quad (6)$$

where $\text{Conv}(E)$ denotes the convex hull of a set E and $\mathbf{B}(0, \mu)$ is the ball centered in 0 and of radius μ.

Proof: Since $S = \text{Conv}(\{v_0, \dots, v_n\})$ and the vector field is affine, it is quite simple to show that

$$\{Ax + a|\ x \in S\} = \text{Conv}(\{Av_0 + a, \dots, Av_n + a\}).$$

Then, the theorem is proved by considering point (i) and (iii) of Proposition 4.1. ■

Following [11], we now establish conditions disabling the exit through a set of facets. The *exit set* of the facet F_j of S is defined as the set

$$\mathrm{cl}(\{x \in F_j|\ m_j^T(Ax + a) > -\mu\}).$$

where $\mathrm{cl}(E)$ denotes the closure of a set E. The facet is said to be *blocked* if its exit set is empty. It is clear that F is blocked if and only if

$$\forall x \in F_j,\ m_j^T(Ax + a) \leq -\mu.$$

Lemma 4.3: If a trajectory of (5), starting in S, exits S at the point x, then x is in the exit set of one of the facets of S.

Proof: The proof is directly adapted from [11]. Let $\mathbf{x}$ be a trajectory of (5), exiting S at the point x (*i.e.* $\mathbf{x}(T) = x$). Let us assume that x is not in the exit set of any facet of S. Let $\mathcal{J} \subset \{0, \ldots, n\}$ be the index set such that $x \in F_j$ iff $j \in \mathcal{J}$. Then, there exists $\delta > 0$ such that

$$\forall j \in \mathcal{J}, \forall z \in F_j,\ \|z - x\| < \delta \implies m_j^T(Ax + a) \leq -\mu \tag{7}$$

and

$$\forall j \in \{0, \ldots, n\} \setminus \mathcal{J},\ \mathrm{dist}(x, F_j) > \delta. \tag{8}$$

Let y be a point in the interior of S, let us define the map $g(z) = (y - z)/(1 + \|y - z\|)$. For all $j \in \{0, \ldots, N\}$, for all $z \in F_j$, $m_j^T g(z) < 0$. Let $\varepsilon \geq 0$ and $\mathbf{z}_\varepsilon$ denote the solution of the differential equation

$$\dot{\mathbf{z}}_\varepsilon(t) = A\mathbf{z}_\varepsilon(t) + a + \varepsilon g(\mathbf{z}_\varepsilon(t)) + \mathbf{d}(t),\ \mathbf{z}_\varepsilon(T) = x.$$

Choose $t_1, \varepsilon_1 > 0$ such that $\|\mathbf{z}_\varepsilon(t) - x\| < \delta$ for any $t \in [T, t_1)$, $\varepsilon \in [0, \varepsilon_1)$. Let $\varepsilon \in (0, \varepsilon_1)$ and assume that $\mathbf{z}_\varepsilon$ exits S at a time $T_1 \in [T, t_1)$, then from (8) it necessarily exits through a facet F_j with $j \in \mathcal{J}$. Let us remark that for all $j \in \mathcal{J}$, such that $\mathbf{z}_\varepsilon(T_1) \in F_j$, (7) implies that

$$\lim_{t \to T_1^+} m_j^T \dot{\mathbf{z}}_\varepsilon(t) \leq -\mu + \varepsilon m_j^T g(\mathbf{z}_\varepsilon(T_1)) + \lim_{t \to T_1^+} m_j^T \mathbf{d}(t) < 0$$

which contradicts the fact that $\mathbf{z}_\varepsilon$ exits S at time T_1. Then, for all $t \in [T, t_1)$, for all $\varepsilon \in (0, \varepsilon_1)$, $\mathbf{z}_\varepsilon(t) \in S$. Since $\|g(z)\|$ is uniformly bounded on $\mathbb{R}^n$, it follows that $\mathbf{z}_\varepsilon(t)$ is continuous in ε, and since S is closed, $\mathbf{z}_\varepsilon(t) \in S$ for all $t \in [T, t_1)$, for all $\varepsilon \in [0, \varepsilon_1)$. Particularly, for $\varepsilon = 0$, this means that $\mathbf{x}(t) \in S$ for all $t \in [T, t_1)$ which contradicts the fact that $\mathbf{x}$ exits S at time T. ■

From the previous lemma, it is meaningful to say that a trajectory of (5) *exits S through a facet F_j* if and only if it exits S at a point in the exit set of F_j. Then, the following result is straightforward:

Proposition 4.4: There exists a trajectory of (5), starting in S, and exiting S through facet F_j if and only if F_j is not a blocked facet.

Proof: If F_j is blocked then its exit set is empty, thus no trajectory can exit through F_j. If F_j is not blocked, then there exists $x \in F_j$, such that $m_j^T(Ax + a) > -\mu$. Consider

the trajectory $\mathbf{x}$ with initial state x and for the constant disturbance $\mathbf{d}(t) = \mu m_j$. Then,

$$m_j^T \dot{\mathbf{x}}(0) = m_j^T(Ax + a) + \mu > 0$$

which means that the trajectory exits through facet F_j. ■

Similar to Proposition 4.1, the previous property can be characterized using the value of the vector field at the vertices of the simplex S:

Theorem 4.5: There exists a trajectory of (5), starting in S, and exiting S through facet F_j if and only if there exists $i \in \{0, \ldots, n\}$, $i \neq j$, such that $m_j^T(Av_i + a) > -\mu$.

Proof: Let us assume that F_j is not a blocked facet, then there exists $x \in F_j$ such that $m_j^T(Ax + a) > -\mu$. x is a convex combination of $\{v_i|\ i \in \{0, \ldots, n\}, i \neq j\}$ and since the vector field is affine, it follows that $m_j^T(Ax + a)$ is a convex combination of $\{m_j^T(Av_i + a)|\ i \in \{0, \ldots, n\}, i \neq j\}$ Thus, there must be at least one vertex v_i of F_j such that $m_j^T(Av_i + a) > -\mu$. The converse implication is obvious. Proposition 4.4 allows us to conclude. ■

B. Controller synthesis

We can now move to the synthesis problem for affine control systems with disturbances on simplices. We essentially consider two types of local controllers on the simplex S, each type of controller solving one of the following problems.

Problem 4.6 (Stay in a simplex): Design an affine feedback controller $h : S \to U$, $h(x) = Kx + k$, such that for all $x_0 \in S$, for all piecewise continuous disturbances $\mathbf{d} : \mathbb{R}^+ \to \mathbb{R}^n$ satisfying $\|\mathbf{d}(t)\| \leq \mu$ for all $t \in \mathbb{R}^+$, the trajectory $\mathbf{x} : \mathbb{R}^+ \to \mathbb{R}^n$ of the closed loop affine system with disturbances

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bh(\mathbf{x}(t)) + a + \mathbf{d}(t),\ \mathbf{x}(0) = x_0$$

satisfies $\mathbf{x}(t) \in S$ for all $t \in \mathbb{R}^+$.

Problem 4.7 (Exit through a set a facets): Consider a subset of indices $\mathcal{I} \subseteq \{0, \ldots, n\}$, and the associated subset of facets $\mathcal{F} = \{F_i|,\ i \in \mathcal{I}\}$, design an affine feedback controller $h : S \to U$, $h(x) = Kx + k$, such that for all $x_0 \in S$, for all piecewise continuous disturbances $\mathbf{d} : \mathbb{R}^+ \to \mathbb{R}^n$ satisfying $\|\mathbf{d}(t)\| \leq \mu$ for all $t \in \mathbb{R}^+$, the trajectory $\mathbf{x} : \mathbb{R}^+ \to \mathbb{R}^n$ of the closed loop affine system with disturbances

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bh(\mathbf{x}(t)) + a + \mathbf{d}(t),\ \mathbf{x}(0) = x_0$$

exits S in finite time through a facet in $\mathcal{F}$.

We denote $u_0, \ldots, u_n \in U$ the values of the controller at the vertices of S:

$$u_i = h(v_i) = Kv_i + k,\ i \in \{0, \ldots, n\}.$$

Since $v_0, \ldots, v_n$ are affinely independent, $u_0, \ldots, u_n$ uniquely determines the matrix K and the vector k. At the vertices of the simplex S, the value of the vector field of the closed loop affine system is given by

$$Av_i + Bh(v_i) + a = Av_i + Bu_i + a.$$

In the following we characterize suitable values of $u_0, \ldots, u_n$ with the understanding that these allows the computation of the affine controller h.

Theorem 4.8: An affine feedback controller h solves problem 4.6 if and only if for all $j \in \{0, \ldots, n\}$, $u_j \in U$ and

$$\forall i, j \in \{0, \ldots, n\}, \ i \neq j, \ m_i^T(Av_j + Bu_j + a) \leq -\mu.$$

Proof: All the trajectories of the closed loop affine system remain in the simplex S forever if and only if there does not exist any trajectory leaving S through one of the facets. The result then follows from Theorem 4.5. ∎

Theorem 4.9: An affine feedback controller h solves problem 4.7 if and only if for all $j \in \{0, \ldots, n\}$, $u_j \in U$ and

$$\forall i \in \{0, \ldots, n\} \setminus \mathcal{I}, \ \forall j \in \{0, \ldots, n\}, \ j \neq i,$$
$$m_i^T(Av_j + Bu_j + a) \leq -\mu$$

and

$$\mathrm{Conv}(\{Av_0 + Bu_0 + a, \ldots, Av_n + Bu_n + a\}) \cap \mathbf{B}(0, \mu) = \emptyset.$$

Proof: From Theorem 4.2, all the trajectories of the closed loop affine system exits the simplex S if and only if the second equation holds. Then, all the trajectories exit through a facet in $\mathcal{F}$ if there does not exist any trajectory leaving S through one of the facets not in $\mathcal{F}$ which is equivalent to the first equation of Theorem 4.5. ∎

We now discuss the computation of input values $u_0, \ldots, u_n$ satisfying the conditions of the previous theorems. Let us remark that for solving problem 4.6, the characterization given by Theorem 4.8 only involves a set of linear inequalities because U is a polytope. Thus, $u_0, \ldots, u_n$ can be computed using linear programming.

The computation of $u_0, \ldots, u_n$ solving problem 4.7 is a little bit more complex. Let us define the following convex polytopes:

$$U_j = \left\{ u \in U \ \middle| \ \begin{array}{l} m_i^T(Av_j + Bu + a) \leq -\mu, \\ \forall i \in \{0, \ldots, n\} \setminus (\mathcal{I} \cup \{j\}) \end{array} \right\}$$

U_j denotes the set of values of u_j such that the first equation of Theorem 4.9 holds. We denote by W_j the set of vertices of U_j.

Proposition 4.10: There exists an affine controller h solving problem 4.7 if and only if there exists $w_0 \in W_0, \ldots, w_n \in W_n$ such that

$$\mathrm{Conv}(\{Av_0 + Bw_0 + a, \ldots, Av_n + Bw_n + a\}) \cap \mathbf{B}(0, \mu) = \emptyset. \tag{9}$$

An admissible controller is given by choosing the input values $u_0 = w_0, \ldots, u_n = w_n$.

Proof: The proof is adapted from [11]. It is clear that the conditions of Proposition 4.10 imply those of Theorem 4.9. Let us show that the converse holds as well. Let us assume that the conditions of Theorem 4.9 hold. Then, there exists $u_0 \in U_0, \ldots, u_n \in U_n$ such that

$$\mathrm{Conv}(\{Av_0 + Bu_0 + a, \ldots, Av_n + Bu_n + a\}) \cap \mathbf{B}(0, \mu) = \emptyset.$$

The intersection of these two convex sets is empty, then from the Separation Theorem (see *e.g.* [6], page 111) there exist $\xi \in \mathbb{R}^n$ and $\gamma > 0$ such that the hyperplane $\xi^T x = \gamma$ separates the two sets. Then,

$$\forall x \in \mathrm{Conv}(\{Av_0 + Bu_0 + a, \ldots, Av_n + Bu_n + a\}), \ \xi^T x > \gamma.$$

Then, for all $j \in \{0, \ldots, n\}$, $\xi^T(Av_j + Bu_j + a) > \gamma$. Since $u_j \in U_j$, u_j is a convex combination of the elements of W_j. It follows, that there exists at least one $w_j \in W_j$ such that $\xi^T(Av_j + Bw_j + a) > \gamma$. It follows that

$$\forall x \in \mathrm{Conv}(\{Av_0 + Bw_0 + a, \ldots, Av_n + Bw_n + a\}), \ \xi^T x > \gamma$$

and therefore (9) holds. ∎

Hence, a controller solving problem 4.7 can be synthesized by computing the vertices of the polytopes $U_0, \ldots, U_n$ and then looking for a suitable combination of vertices.

In the following section, we show how we can use these robust local controllers to solve algorithmically the motion planning problem 3.4.

V. CONTROL OF THE HYBRIDIZATION

We now briefly describe the approach for solving the motion planning problem 3.4 for the piecewise affine hybrid system on simplices with disturbances Σ'.

Let $\mathcal{S}'$ be a subset of the triangulation $\mathcal{S}$ and let $S \in \mathcal{S}$, we denote by *common-facets*$(S, \mathcal{S}')$ the subset of facets of the simplex S that are also facets of a simplex in $\mathcal{S}'$. We denote by *adjacent*$(\mathcal{S}')$ the subset of simplices that are adjacent to a simplex in $\mathcal{S}'$: $S \in$ *adjacent*$(\mathcal{S}')$ if and only if $S \in \mathcal{S} \setminus \mathcal{S}'$ and *common-facets*$(S, \mathcal{S}') \neq \emptyset$.

We start by synthesizing a controller that keep the trajectories of Σ' in the set of final states F:

Algorithm 5.1:
1) Set $\mathcal{S}' := \emptyset$;
2) For $S_i \in \mathcal{S}_F$, loop: if problem 4.6 is solvable with $(S, A, B, a, \mu) = (S_i, A_i, B_i, a_i, \mu_i)$, by local affine controller h_i then set $\mathcal{S}' := \mathcal{S}' \cup \{S_i\}$ and let $h := h_i$ on simplex S_i;
3) Set $F' := \bigcup_{S_i \in \mathcal{S}'} S_i$.

Then, by construction, we obviously have the following property:

Lemma 5.2: Let $F' \subseteq F$ and $h : F' \rightarrow U$ be computed by Algorithm 5.1, then for all trajectories of Σ', starting in F' and given by, when $\mathbf{x}(t) \in S_i$,

$$\dot{\mathbf{x}}(t) = A_i \mathbf{x}(t) + B_i h(\mathbf{x}(t)) + a_i + \mathbf{d}(t), \ \|\mathbf{d}(t)\| \leq \mu_i,$$

the following holds: for all $t \in R^+$, $\mathbf{x}(t) \in F$.

Remark 5.3: Algorithm 5.1 ensures that the trajectories of Σ' entering a simplex in $\mathcal{S}'$ remains in this simplex forever. One could actually relax this condition: the trajectories entering a simplex in $\mathcal{S}'$ either remain in this simplex or move to another simplex in $\mathcal{S}'$. A controller satisfying this specification can be easily computed using local controllers blocking a subset of facets of the simplices and an iterative fix-point procedure to compute the invariant subset F'. However, this approach, contrarily to the one presented in the paper, does not exclude *Zeno* behaviors easily and requires further work.

We then synthesize a controller that drives the trajectories of Σ' in the set F' computed by Algorithm 5.1.

Algorithm 5.4:

1) Let $\mathcal{S}'$, h, be computed by Algorithm 5.1;
2) While $\mathcal{S}_I \not\subseteq \mathcal{S}'$, loop:
   - If there exists $S \in adjacent(\mathcal{S}')$ such that problem 4.7 is solvable with $(S, A, B, a, \mu, \mathcal{F}) = (S_i, A_i, B_i, a_i, \mu_i, common\text{-}facets(S, \mathcal{S}'))$, by local affine controller h_i then set $\mathcal{S}' := \mathcal{S}' \cup \{S_i\}$ and let $h := h_i$ on simplex S_i;
   - Else, exit;
3) Set $D' := \bigcup_{S_i \in \mathcal{S}'} S_i$.

Lemma 5.5: Let $D' \subseteq D$ and $h : D' \to U$ be computed by Algorithm 5.4, then for all trajectories of Σ', starting in D' and given by, when $\mathbf{x}(t) \in S_i$,

$$\dot{\mathbf{x}}(t) = A_i \mathbf{x}(t) + B_i h(\mathbf{x}(t)) + a_i + \mathbf{d}(t), \ \|\mathbf{d}(t)\| \leq \mu_i,$$

the following holds:

$$\forall t \in \mathbb{R}^+, \ \mathbf{x}(t) \in D' \text{ and } \exists T \in \mathbb{R}^+, \ \forall t \geq T, \ \mathbf{x}(t) \in F.$$

Proof: By construction of the controller h, when a trajectory exits a simplex in $\mathcal{S}'$, it necessarily exits through a facet that is common with another simplex in $\mathcal{S}'$. Then, it is straightforward that all the trajectories satisfy $\mathbf{x}(t) \in D'$, for all $t \in \mathbb{R}^+$. Let us remark that the Algorithm 5.4 stops after $K < N$ iterations. Let $\mathcal{S}'_k$ denote the value of the variable $\mathcal{S}'$ after the kth iteration; $\mathcal{S}'_0$ being the set of simplices computed by Algorithm 5.1. Let us show that $\mathbf{x}$ reaches a simplex in $\mathcal{S}'_0$ in finite time. Initially, $\mathbf{x}(0) \in D'$ and thus belongs to a simplex in $\mathcal{S}'_K$. Assume that for some $k \in \{1, \dots, K\}$, and some $t \in \mathbb{R}^+$, $\mathbf{x}(t)$ belongs to a simplex $S \in \mathcal{S}'_k$. Let $k' < k$ be the smallest integer such that $S \notin \mathcal{S}'_{k'}$ and $S \in \mathcal{S}'_{k'+1}$. This essentially means that $\mathcal{S}'_{k'+1} = \mathcal{S}'_{k'} \cup \{S\}$. By construction of the controller h, the trajectory $\mathbf{x}$ leaves S at a time $t' \geq t$ through a facet that is common with a simplex in $\mathcal{S}'_{k'}$. Hence, $\mathbf{x}(t')$ belongs to a simplex in $\mathcal{S}'_{k'}$. Since $k' < k$, this ensures, by induction, that $\mathbf{x}$ reaches a simplex in $\mathcal{S}'_0$ in finite time: there exists $T \in \mathbb{R}^+$, such that $\mathbf{x}(T) \in F'$. Then, by Lemma 5.2, it follows that for all $t \geq T$, $\mathbf{x}(t) \in F$. ∎

Finally, we have, as a straightforward consequence of the previous lemma:

Theorem 5.6: Let $D' \subseteq D$ and $h : D' \to U$ be computed by Algorithm 5.4, if $I \subseteq D'$, then h solves the motion planning problems 2.1 and 3.4.

Let us remark that our approach for solving the motion planning problem 2.1 is clearly conservative. Our algorithm may fail to solve the problem even though a suitable controller exists. There are several sources of conservatism. The first one is due to the use of a hybridization. Proposition 3.6 suggests that this can be reduced by using a finer triangulation at the price of an increased computational effort. The other sources of conservatism are inherent to the approach developed in [11]. However, this conservatism allows us to synthesize controllers that are correct by design by a fully automated method which is computationally effective as shown is the following section.
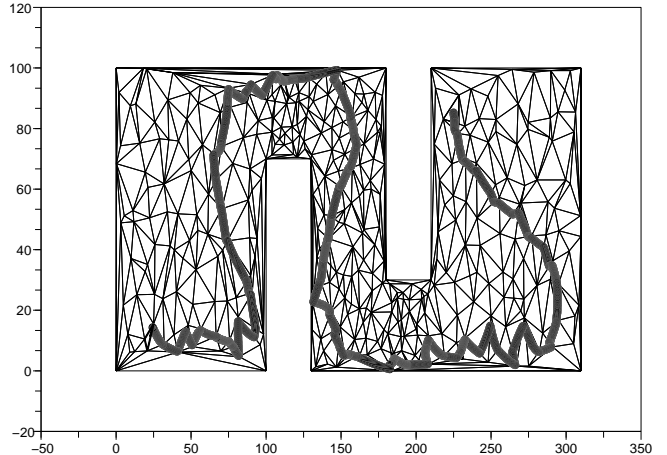

Fig. 1. Triangulation of the state domain and a trajectory of the controlled robot swimming in running water. The robot starts in the upper right region and reaches the bottom left region while remaining in the state domain. It has to swim against the water flow.

VI. EXAMPLE

We now consider the following problem. Let $\mathbf{x}(t) = (\mathbf{x}_1(t), \mathbf{x}_2(t))$ denote the position of an autonomous swimming robot evolving in running water. The dynamics of the system is given by:

$$\begin{cases} \dot{\mathbf{x}}_1(t) &= \sin\left(\frac{\pi}{w}\mathbf{x}_2(t)\right) + \mathbf{u}_1(t) &, \ \mathbf{u}_1(t) \in [-\alpha, \alpha] \\ \dot{\mathbf{x}}_2(t) &= \mathbf{u}_2(t) &, \ \mathbf{u}_2(t) \in [-\alpha, \alpha] \end{cases}$$

where w is the width of the river. At the center of the river, the water speed is high while it is zero near the riverbank.

The sets of initial and final states I and F are such that the robot has to swim against the water flow to reach F. The nonconvex state domain D can be seen on Figure 1. The motion planning problem is trivial if $\alpha > 1$. In the following, we considered $\alpha = 0.83$.

We applied our approach to this motion planning problem. On Figure 1, we can see the triangulation used by our algorithm and a trajectory of the controlled robot. The robot starts in the upper right region and reaches the bottom left region while remaining in the state domain. We can see that each time the robot crosses the river (from top to bottom or from bottom to top on Figure 1), the water flow pushes the robot towards the right. Hence, the robot first has to move to the left, in order to cross safely the river. This is clearly seen on Figure 1.

VII. CONCLUSION

In this paper, we presented an algorithmic approach to a motion planning problem for nonlinear systems. Our technique is based on two main ingredients, namely a hybridization and robust controllers on simplices. Though conservative, our method can be fully automated and we showed that it is effective on an example. Our method should probably be reserved to small-dimensional systems as the number of simplices in the triangulation explodes when the dimension grows.

There are several possible extensions for this work. First, instead of using piecewise affine hybridization defined on a triangulation, one could use piecewise multi-affine hybridization [2] defined on a partition of the state domain in hypercubes as an extension of [4] would allow us to synthesize local controllers on hypercubes. Second, by extending the approach presented in [9], [15], one could solve more complex motion planning problems such as those specified in linear temporal logic (LTL). Finally, the class of nonlinear system we consider could be extended, by considering systems of the form

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)), \ \mathbf{x}(t) \in D, \ \mathbf{u}(t) \in U.$$

Following [8], a piecewise affine hybridization could be computed by triangulating the domain $D \times U$.

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